8. Partial Differential Equations

Consistency: difference equation $\rightarrow$ differential equation as $dt \rightarrow 0$ (truncation error $\rightarrow 0$ as $dt \rightarrow 0$)?

Stability: computed solution of the difference equation $\approx$ exact solution of the difference (rounding error under control?)

Convergence: computed solution to the difference equation $\rightarrow$ exact solution to the differential equation as $dt \rightarrow 0$?

Lax Equivalence Theorem

Given a properly posed linear initial value problem and a finite difference approximation to it that satisfies the consistency condition, stability is necessary and sufficient condition for convergence.

A general linear 2nd-order 2-variable PDE:

$$a(x,y)\frac{\partial^2 u}{\partial x^2}+b(x,y)\frac{\partial^2 u}{\partial x\partial y}+c(x,y)\frac{\partial^2 u}{\partial y^2}$$
$$+d(x,y)\frac{\partial u}{\partial x}+e(x,y)\frac{\partial u}{\partial y}+f(x,y)u=g(x,y)$$

$b^2-4ac>0$: hyperbolic PDE (two characteristic curves)
~ wave equation

$b^2-4ac=0$: parabolic PDE (one characteristic curve)
~ heat diffusion equation

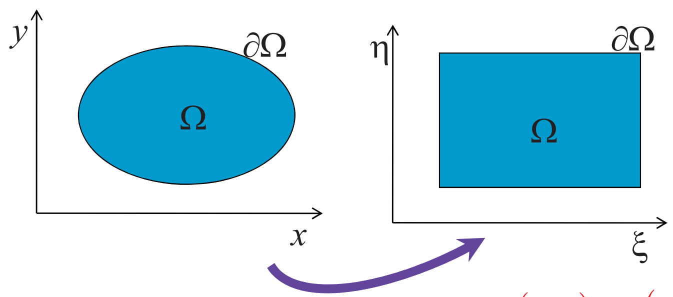
$b^2-4ac<0$: elliptic PDE (no characteristic curve)
~ Laplace equation

§ elliptic PDE --- Laplace (Poisson) equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=g(x,y) \quad \text{in some region } \Omega$$

~ no characteristic curve

B.C. $T(x,y)=f(x,y)$ on the boundary $\partial\Omega$



$$\begin{cases} x=x(\xi,\eta) \\ y=y(\xi,\eta) \end{cases}$$

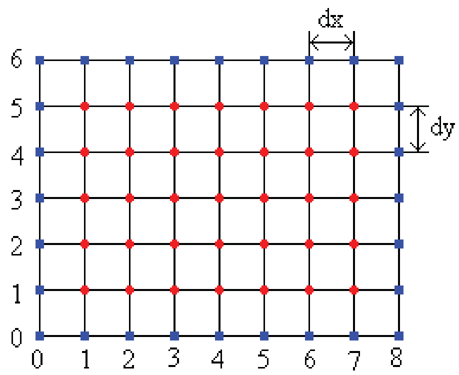
$$T(x,y)=T\big(x(\xi,\eta),y(\xi,\eta)\big)=\tilde{T}(\xi,\eta)$$

§ Finite difference solutions

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = g(x, y) \quad \text{in some region } \Omega$$

$$\text{B.C.} \quad T(x, y) = f(x, y) \quad \text{on the boundary } \partial\Omega$$

Step 1: mesh the interested domain $a \le x \le b$ and $c \le y \le d$



Enforce equality at each grid point:

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_{i,j} + \left(\frac{\partial^2 T}{\partial y^2}\right)_{i,j} = g(x_i, y_j) = g_{i,j}$$

$$\text{for} \quad i = 1, 2, \cdots, N_x - 1 \quad \text{and} \quad j = 1, 2, \cdots, N_y$$

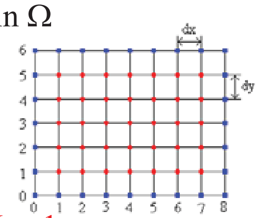
Step 2: do numerical differentiation, e.g. central difference

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_{i,j} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} + O(\Delta x^2)$$

$$\left(\frac{\partial^2 T}{\partial y^2}\right)_{i,j} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2} + O(\Delta y^2)$$

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2} = g_{i,j}$$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=g(x,y)$$ ~ true for any point (x, y) in Ω



$$\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\Delta x^2}+\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\Delta y^2}=g_{i,j}$$

enforced for $i=1,2,\cdots,N_x-1$ and $j=1,2,\cdots,N_y-1$

- take the solution of the difference equations as an approximation to the solution of the PDE.
- truncation error ~ $O(\Delta x^2, \Delta y^2)$
- coupled difference equations for $(N_x-1)(N_y-1)$ unknowns.
- boundary conditions: $T_{0,j}$ for $j=0,1,2,\ldots,N_y$ and
  $T_{i,0}$ for $i=0,1,2,\ldots,N_x$

§ consistency:

$$g_{i,j}=\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\Delta x^2}+\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\Delta y^2}$$

$$=\left(\frac{\partial^2 T}{\partial x^2}\right)_{i,j}+\left(\frac{\partial^2 T}{\partial y^2}\right)_{i,j}+\frac{\Delta x^2}{12}\left(\frac{\partial^4 T}{\partial x^4}\right)_{i,j}+\frac{\Delta y^2}{12}\left(\frac{\partial^4 T}{\partial y^4}\right)_{i,j}+O\left(\Delta x^4,\Delta y^4\right)$$

$$\rightarrow\left(\frac{\partial^2 T}{\partial x^2}\right)_{i,j}+\left(\frac{\partial^2 T}{\partial y^2}\right)_{i,j}\quad \text{as } \Delta x,\Delta y\rightarrow 0$$

- For small but finite Δx and Δy, the difference equation approximates better the following PDE (called the modified equation):

$$g=\underbrace{\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}}_{\text{physical diffusion}}+\underbrace{\frac{\Delta x^2}{12}\frac{\partial^4 T}{\partial x^4}+\frac{\Delta y^2}{12}\frac{\partial^4 T}{\partial y^4}}_{\text{numerical diffusion}}$$

§ stability: ok because there is no time marching

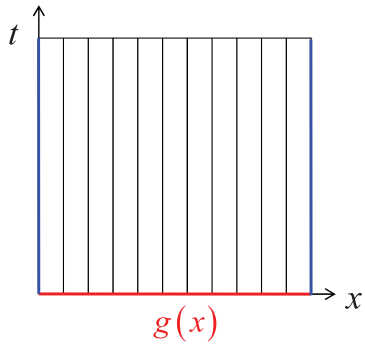
§ Parabolic PDE --- one characteristic curve

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} = f(x,t)$$

IC: $u(x,0) = g(x)$

BC: $u(a,t) = p(t)$ and $u(b,t) = q(t)$

- viewed as a time marching problem



- spatial discretization

  $i = 0,1,2,\ldots, N$

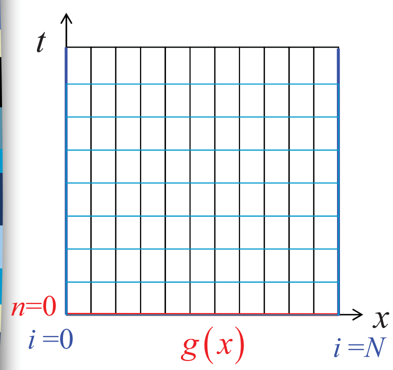
$$\begin{cases} u(a,t) = u_0(t) = p(t) \\ u(b,t) = u_N(t) = q(t) \end{cases}$$

- Choose central difference for the spatial derivative

$$\left(\frac{\partial u}{\partial t}\right)_i = f(x_i,t) = \alpha \left(\frac{\partial^2 u}{\partial x^2}\right)_i \approx \alpha \cdot \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$$

$\sim N\!-\!1$ ODEs for $i = 1,2,\ldots,N\!-\!1$

- Choose a time marching scheme, say forward Euler method



$$u_i^n \equiv u(x_i,t_n)$$

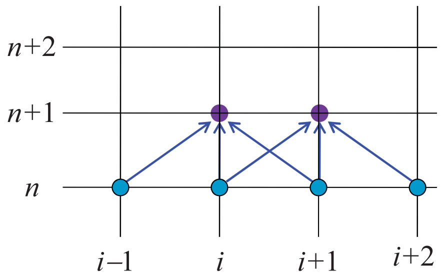
want: u_i^n $\begin{cases} \text{for } i = 1,2,\cdots,N-1 \\ \text{and } n = 1,2,3,\cdots \end{cases}$

$$u_i^{n+1} = u_i^n + \Delta t \cdot f_i^n$$

$$= u_i^n + \Delta t \cdot \alpha \cdot \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

- central difference + forward Euler method

$$u_i^{n+1} = u_i^n + \frac{\alpha \Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$



§ consistency: $\dfrac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha\left(\dfrac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}\right)$

$$\left(\frac{\partial u}{\partial t}\right)_i^n + \frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_i^n + O\left(\Delta t^2\right)$$

$$= \alpha\left(\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n + \frac{\Delta x^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_i^n + O\left(\Delta x^4\right)\right)$$

$$\rightarrow \frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \quad \text{as} \quad \Delta t, \Delta x \rightarrow 0$$

- modified equation: $\dfrac{\partial u}{\partial t} + \dfrac{\Delta t}{2}\dfrac{\partial^2 u}{\partial t^2} = \alpha \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\alpha \Delta x^2}{12}\dfrac{\partial^4 u}{\partial x^4}$

- modified equation: $\frac{\partial u}{\partial t}+\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}=\alpha\frac{\partial^2 u}{\partial x^2}+\frac{\alpha\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4}$

$$\frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial t^2}=\alpha\frac{\partial}{\partial t}\frac{\partial^2 u}{\partial x^2}=\alpha\frac{\partial^2}{\partial x^2}\frac{\partial u}{\partial t}=\alpha\frac{\partial^2}{\partial x^2}\left(\alpha\frac{\partial^2 u}{\partial x^2}\right)=\alpha^2\frac{\partial^4 u}{\partial x^4}$$

modified equation: $\frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2}+\frac{\alpha\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4}-\frac{\alpha^2\Delta t}{2}\frac{\partial^4 u}{\partial x^4}$

$$\frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2}-\underbrace{\left\{\frac{\alpha^2\Delta t}{2}-\frac{\alpha\Delta x^2}{12}\right\}}_{\text{hyper-viscosity}}\frac{\partial^4 u}{\partial x^4}$$

- physical viscosity > 0 always
- numerical viscosity may be negative!

§ diffusion --- spatial derivatives of even order

$$\frac{\partial u}{\partial t}=D_{2n}\frac{\partial^{2n} u}{\partial x^{2n}}$$

Write $u(x,t)=A(t)\exp(ikx)$. Then

$$e^{ikx}\frac{dA}{dt}=D_{2n}\cdot(ik)^{2n}Ae^{ikx}=(-1)^n D_{2n}\cdot k^{2n}Ae^{ikx}$$

$$\frac{dA}{dt}=(-1)^n\cdot D_{2n}\cdot k^{2n}A$$

$$A(t)=A(0)\exp\left\{D_{2n}(-1)^n k^{2n}t\right\}$$

$$u(x,t)=A(0)\exp\left\{D_{2n}(-1)^n k^{2n}t\right\}\exp(ikx)$$

Amplitude $\begin{cases}\text{decreases}\\ \text{increases}\end{cases}$ with time if $\begin{cases}(-1)^n D_{2n}<0\\ (-1)^n D_{2n}>0\end{cases}$ $\begin{cases}D_2>0, D_4<0\\ D_2<0, D_4>0\end{cases}$

§ dispersion --- spatial derivatives of odd order

$$\frac{\partial u}{\partial t} = D\frac{\partial^{2n+1} u}{\partial x^{2n+1}}$$

Write $u(x,t) = A(t)\exp(ikx)$. Then

$$\frac{dA}{dt} = D(ik)^{2n+1} A \;= (-1)^n\, i\, Dk^{2n+1} A$$

$$A(t) = A(0)\exp\left\{(-1)^n\, i\, Dk^{2n+1} t\right\}$$

$$u(x,t) = A(0)\exp\left\{i\left(kx + (-1)^n\; Dk^{2n+1} t\right)\right\}$$

- Amplitude remains unchanged with time.
- Phase changes with time.
- In general, the phase change is different for different *k*.

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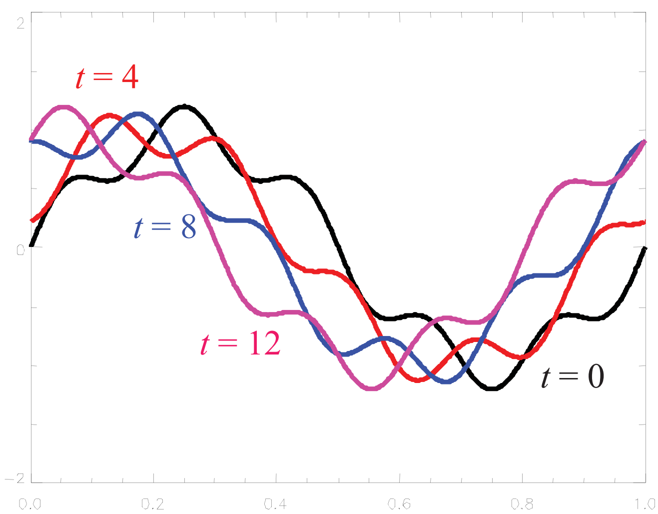
$$u(x,t) = \sin(2\pi x + 0.1t) + 0.2\,\sin(10\pi x + 100t)$$



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§ time-marching $\Rightarrow$ accumulation of rounding error
$\Rightarrow$ stability problem

§ stability (Von Neumann method)

Consider $u(x,t)$ as a combination of modes of different wave numbers,

$$u(x,t)=\int_{-\infty}^{\infty} A(k,t)\exp(ikx)\,dk$$

where $A(k,t)$ is the amplitude of the mode of wave number k.

- If $A(k,t)$ diverges (grows exponentially) for any k, then $u(x,t)$ diverges.
- To have a stable scheme (non-divergent results), amplitudes of all modes must remain bounded at all times.
- We examine the stability of each mode (k) one by one.

- central difference + forward Euler method for the diffusion eqn.:

$$u_i^{n+1}=u_i^n+\frac{\alpha\Delta t}{\Delta x^2}\left(u_{i+1}^n-2u_i^n+u_{i-1}^n\right)$$

Single-mode analysis:

$$u_i^n=u(x_i,t_n)=A(k,t_n)\exp(ikx_i)\equiv C\rho^n\exp(ikx_i)$$

$$C\rho^{n+1}e^{ikx_i}=C\rho^n e^{ikx_i}+\frac{\alpha\Delta t}{\Delta x^2}\cdot C\rho^n\cdot\left(e^{ikx_{i+1}}-2e^{ikx_i}+e^{ikx_{i-1}}\right)$$

$$\rho=1+\frac{\alpha\Delta t}{\Delta x^2}\cdot\left(e^{ik\Delta x}-2+e^{-ik\Delta x}\right)=1+\frac{\alpha\Delta t}{\Delta x^2}\cdot\left(2\cos(k\Delta x)-2\right)$$

where $x_{i+1}-x_i=x_i-x_{i-1}=\Delta x$

$$\rho=1+\frac{2\alpha\Delta t}{\Delta x^2}\cdot\left(\cos(k\Delta x)-1\right)$$

stability requires $-1<\ 1-\frac{4\alpha\Delta t}{\Delta x^2}\le\rho\le 1$ for arbitrary k

Remark:
The scheme (forward Eular in time and central difference in space) is stable if

$$-1 < 1 - \frac{4\alpha\Delta t}{\Delta x^2}$$

$$\text{or} \quad \alpha\frac{\Delta t}{\Delta x^2} < \frac{1}{2}$$

~ Courant-Friedrichs-Lewy (CFL) condition ~

How about other time-marching schemes?

$$\left(\frac{\partial u}{\partial t}\right)_i = f(x_i, t) = \alpha\left(\frac{\partial^2 u}{\partial x^2}\right)_i \approx \alpha\cdot\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$$

Single-mode analysis:

$$u_i^n = u(x_i, t_n) = A(k, t_n)\exp(ikx_i) \equiv C\rho^n \exp(ikx_i)$$

$$u_i = u(x_i, t) = A(k, t)\exp(ikx_i)$$

$$\frac{dA}{dt} = \frac{\alpha}{\Delta x^2}A\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)$$

$$= \frac{2\alpha}{\Delta x^2}\left(\cos(k\Delta x) - 1\right)A \equiv \lambda A$$

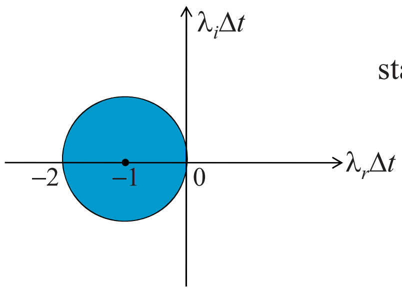
$$\Rightarrow \quad \lambda\Delta t = \frac{2\alpha\Delta t}{\Delta x^2}\left(\cos(k\Delta x) - 1\right) \quad \text{(real for any } k\text{)}$$

$$\Rightarrow \quad -\frac{4\alpha\Delta t}{\Delta x^2} \le \lambda\Delta t \le 0 \quad \text{for arbitrary } k$$

$$\frac{dy}{dt} = \lambda y$$

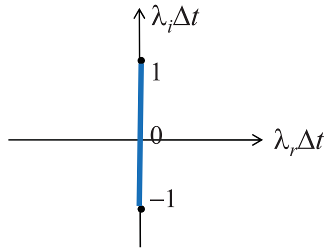
$$-\frac{4\alpha\Delta t}{\Delta x^2} \le \lambda\Delta t \le 0$$

- forward Euler method



stable if $-2 < -\frac{4\alpha\Delta t}{\Delta x^2}$ or $\frac{\alpha\Delta t}{\Delta x^2} < \frac{1}{2}$

- leap frog method

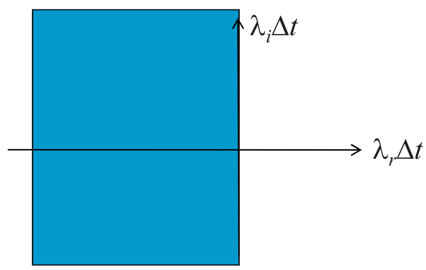


unconditionally
unstable!!

$$\frac{dy}{dt} = \lambda y$$

$$-\frac{4\alpha\Delta t}{\Delta x^2} \le \lambda\Delta t \le 0$$

- Crank-Nicolson method



unconditionally

stable!!

Remark: A time-marching scheme is stable if, given Δx, $\lambda\Delta t$ belongs to the corresponding stability region for all wave numbers k's.

- Dufort Frankel method

leap frog: $\dfrac{u_i^{n+1}-u_i^{n-1}}{2\Delta t}=\alpha\cdot\dfrac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{\Delta x^2}$ (unconditionally stable)

modified: $\dfrac{u_i^{n+1}-u_i^{n-1}}{2\Delta t}=\alpha\cdot\dfrac{u_{i+1}^n-\left(u_i^{n+1}+u_i^{n-1}\right)+u_{i-1}^n}{\Delta x^2}$

~ implicit, two-step, unconditionally stable! ~

§ consistency:

$$\frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2}-\frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3}+\frac{\alpha\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4}+\frac{\alpha\Delta t^2}{\Delta x^2}\frac{\partial^2 u}{\partial t^2}+O\left(\Delta t^4,\Delta x^4,\Delta t^4/\Delta x^2\right)$$

$$\to\alpha\frac{\partial^2 u}{\partial x^2}\quad\text{as}\quad\Delta t,\Delta x\to 0\,?$$

yes only if $\lim\limits_{\Delta t,\Delta x\to 0}\dfrac{\Delta t}{\Delta x}=0$

§ grid testing:

$$\frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2}-\frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3}+\frac{\alpha\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4}+\frac{\alpha\Delta t^2}{\Delta x^2}\frac{\partial^2 u}{\partial t^2}+O\left(\Delta t^4,\Delta x^4,\Delta t^4/\Delta x^2\right)$$

Δt	Δx
0.1	0.1
0.05	0.05
0.01	0.01
0.005	0.005
…	…

$\to\dfrac{\partial u}{\partial t}=\alpha\dfrac{\partial^2 u}{\partial x^2}+\alpha\dfrac{\partial^2 u}{\partial t^2}$

converges to the wrong PDE!

$\Delta t/\Delta x=1/2^m$

Δt	Δx
0.1	0.1
0.025	0.05
0.00625	0.025
0.001	0.008
…	…

$\to\dfrac{\partial u}{\partial t}=\alpha\dfrac{\partial^2 u}{\partial x^2}$

§ hyperbolic PDE --- wave equation

$$0 = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u$$

define $\xi = x - ct$ and $\eta = x + ct$

write $u = u\left(x(\xi,\eta), y(\xi,\eta)\right) = U(\xi,\eta)$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t} = c\left(-\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$

$$\frac{\partial}{\partial t} + c\frac{\partial}{\partial x} = c\left(-\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) + c\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) = 2c\frac{\partial}{\partial \eta}$$

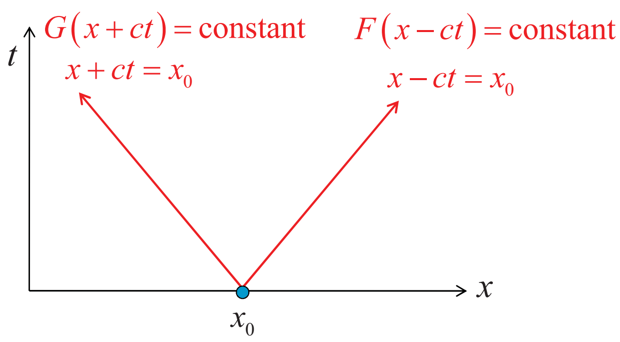
$$\frac{\partial}{\partial t} - c\frac{\partial}{\partial x} = c\left(-\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) - c\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) = -2c\frac{\partial}{\partial \xi}$$

$$\left(\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta}\right)U = 0 \Rightarrow \frac{\partial}{\partial \xi}\left(\frac{\partial U}{\partial \eta}\right) = 0 \Rightarrow \frac{\partial U}{\partial \eta} = \text{fn. of } \eta \text{ only}$$

$$\Rightarrow \frac{\partial}{\partial \eta}\left(\frac{\partial U}{\partial \xi}\right) = 0 \Rightarrow \frac{\partial U}{\partial \xi} = \text{fn. of } \xi \text{ only}$$

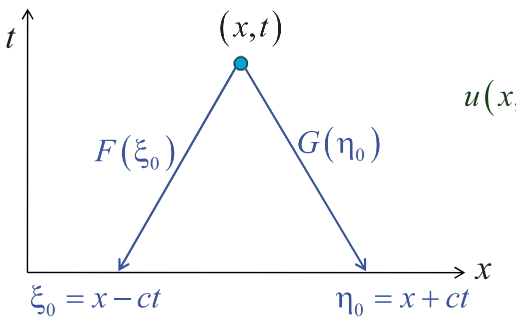
$$U(\xi,\eta) = F(\xi) + G(\eta)$$

$$u(x,y) = F(x-ct) + G(x+ct)$$



wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

characteristic curve	compatibility condition
$x - ct = \text{constant}$	$F(x-ct) = \text{constant}$
$x + ct = \text{constant}$	$G(x+ct) = \text{constant}$



$$u(x,t) = F(x-ct) + G(x+ct)$$
$$= F(\xi_0) + G(\eta_0)$$

§ hyperbolic PDE --- first-order wave equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

$$u(x,0) = f(x) \quad \text{for} \quad -\infty < x < \infty$$

exact solution: $u(x,t) = f(x-ct)$

$$\begin{cases} \text{characteristic curve:} & dx/dt = c \\ \text{compatibility condition:} & u(x,t) = \text{constant} \end{cases}$$

- central difference in space : $\frac{\partial u_i}{\partial t} = -c\frac{u_{i+1} - u_{i-1}}{2\Delta x}$

single-mode analysis: $u_i = A(t)e^{ikx_i}$

$$\frac{dA}{dt} = -\frac{c}{2\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)A$$

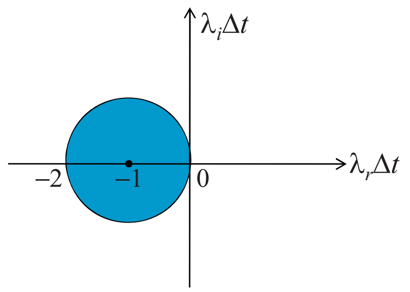
$$\frac{dA}{dt} = -i\frac{c}{\Delta x}\sin(k\Delta x)\cdot A \equiv \lambda A$$

$$\lambda\Delta t = -i\frac{c\Delta t}{\Delta x}\sin(k\Delta x) \qquad \text{(pure imaginary)}$$

$$\lambda_r\Delta t = 0$$

$$-\frac{c\Delta t}{\Delta x} \le \lambda_i\Delta t \le \frac{c\Delta t}{\Delta x}$$

- forward Euler method



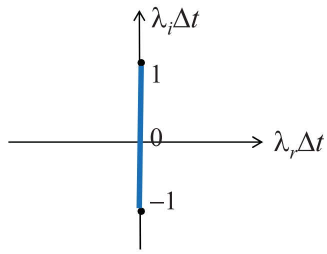
unconditionally
unstable!!

$$\frac{dy}{dt} = \lambda y$$

$$\lambda_r\Delta t = 0$$

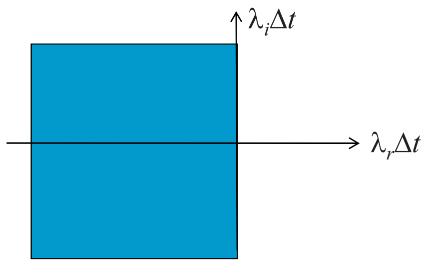
$$-\frac{c\Delta t}{\Delta x} \le \lambda_i\Delta t \le \frac{c\Delta t}{\Delta x}$$

- leap frog method



stable if $\frac{c\Delta t}{\Delta x} < 1$

- Crank-Nicolson method



unconditionally
stable!!

§ consistency: $\dfrac{\partial u_i}{\partial t} = -c\dfrac{u_{i+1}-u_{i-1}}{2\Delta x}$

$$\frac{\partial u_i}{\partial t} = -c\frac{u_{i+1}-u_{i-1}}{2\Delta x} = -c\left\{\left(\frac{\partial u}{\partial x}\right)_i + \frac{\Delta x^2}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + O\left(\Delta x^4\right)\right\}$$

$$\to -c\left(\frac{\partial u}{\partial x}\right)_i \quad \text{as} \quad \Delta x, \Delta t \to 0$$

numerical dispersion

• modified equation :

$$\frac{\partial u}{\partial t} = -c\left(\frac{\partial u}{\partial x}\right) - c\frac{\Delta x^2}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)$$

$$\frac{\partial u_i}{\partial t} = -c\frac{u_{i+1}-u_{i-1}}{2\Delta x} \qquad u_i = A(t)e^{ikx_i}$$

$$\frac{dA}{dt} = -i\frac{c}{\Delta x}\sin\left(k\Delta x\right)\cdot A \equiv \lambda A \quad \Rightarrow \quad A(t) = A(0)\exp\left\{-i\frac{\sin\left(k\Delta x\right)}{\Delta x}\cdot ct\right\}$$

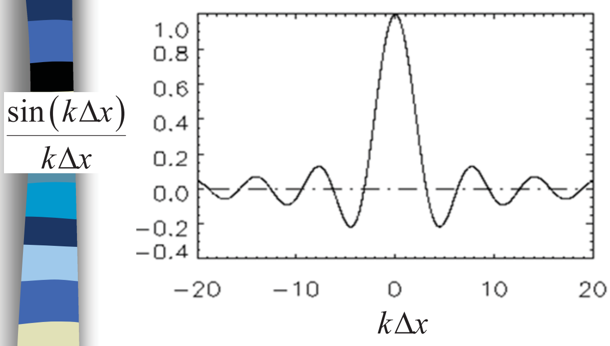
$$\Rightarrow \quad u(x,t) = A(0)\exp\left\{ik\left[x - \frac{\sin\left(k\Delta x\right)}{k\Delta x}\cdot ct\right]\right\}$$

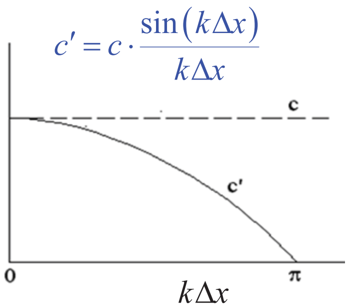
~ phase speed is k-dependent!

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} \qquad \text{IC:} \quad u(x,0) = f(x) = A(0)e^{ikx}$$

$$\Rightarrow \quad u(x,t) = f(x-ct) = A(0)\exp\left\{ik\left(x-ct\right)\right\}$$

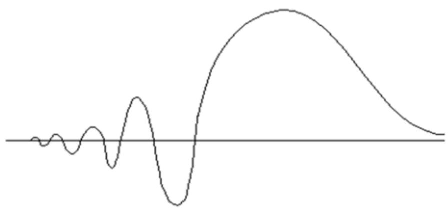
phase speed = c = constant (independent of k)





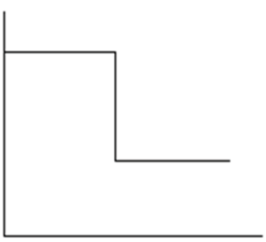
Shorter waves move slower.

earlier time

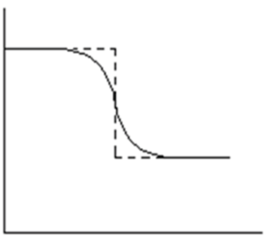


later time

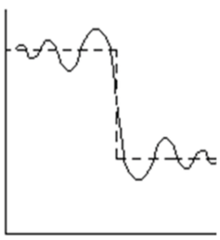
§ Summary: for example, shock wave or any discontinuity



exact



with positive numerical diffusion



with numerical dispersion

- positive numerical diffusion ⇒ more dissipation (energy sink)
- negative numerical diffusion ⇒ energy source ⇒ may blow up eventually if it cannot be balanced by the physical dissipation.
- unstable scheme ⇒ add artificial (explicit) numerical dissipation to obtain stability but sacrifices accuracy.

§ Summary --- error analysis

Suppose the exact solution of the original PDE is

$$u(x,t) = C\exp\{i(kx-\omega t)\} = C\exp\{\omega_i t + i(kx-\omega_r t)\}$$

$$u(x_i,t_n) = Ce^{n\omega_i \Delta t}\exp\{i(kx-n\omega_r \Delta t)\} = C\left(e^{\omega_i \Delta t}\right)^n \exp\{i(kx-n\omega_r \Delta t)\}$$

$$\text{exact}\begin{cases}\text{amplitification factor } = e^{\omega_i \Delta t}\\ \text{phase change } = \omega_r \Delta t\end{cases}$$

write the numerical solution as

$$u(x_i,t_n) = C\rho^n \exp(ikx) = C\left(|\rho| e^{-i\theta}\right)^n \exp(ikx)$$

$$= C|\rho|^n \exp\{i(kx-n\theta)\}$$

$$\text{numerical}\begin{cases}\text{amplitification factor } = |\rho|\\ \text{phase change } = \theta\end{cases}$$

$$\text{exact}\begin{cases}\text{amplitification factor } = e^{\omega_i \Delta t}\\ \text{phase change } = \omega_r \Delta t\end{cases}$$

$$\text{numerical}\begin{cases}\text{amplitification factor } = |\rho|\\ \text{phase change } = \theta\end{cases}$$

- diffusion/dissipation error $\equiv \dfrac{\text{numerical amplication factor}}{\text{exact amplication factor}}$

$$= |\rho| / \exp(\omega_i \Delta t)$$

$$= 1 \quad \Leftrightarrow \quad \text{no dissipation error}$$

- dispersion error $\equiv$ numerical phase change $-$ exact phase change

$$= \theta - \omega_r \Delta t$$

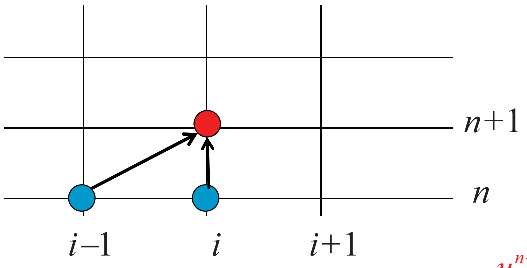
$$= 0 \quad \Leftrightarrow \quad \text{no dispersion error}$$

§ wave equation --- characteristic-curve method

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0$$

- forward Euler in time and backward difference in space (upwind scheme)

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+c\frac{u_i^n-u_{i-1}^n}{\Delta x}=0$$



$$u_i^{n+1}=\left(1-\frac{c\Delta t}{\Delta x}\right)u_i^n+\frac{c\Delta t}{\Delta x}u_{i-1}^n$$

truncation error ~ $O(\Delta t, \Delta x)$

§ modified equation

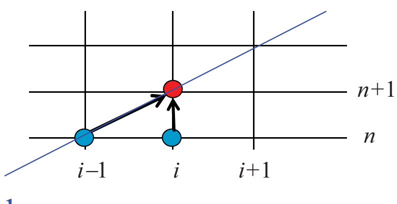
$$\frac{u_i^{n+1}-u_i^n}{dt}+c\frac{u_i^n-u_{i-1}^n}{dx}=0$$

$$\frac{\partial u}{\partial t}=c\frac{\partial u}{\partial x}-\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}+\frac{c\Delta x}{2}\frac{\partial^2 u}{\partial x^2}+O\left(\Delta t^2,\Delta x^2\right)$$

$$\because\ \frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}\ \Rightarrow\ \frac{\partial u}{\partial t}=c\frac{\partial u}{\partial x}+\frac{c\Delta t}{2}\left(\frac{\Delta x}{\Delta t}-c\right)\frac{\partial^2 u}{\partial x^2}+O\left(\Delta t^2,\Delta x^2\right)$$

0 if $\Delta x=c\Delta t$

In fact, truncation error = 0 exactly as $\Delta x = c\,\Delta t$!



char. curve

$$u_i^{n+1}=\left(1-\frac{c\Delta t}{\Delta x}\right)u_i^n+\frac{c\Delta t}{\Delta x}u_{i-1}^n$$

$$=u_{i-1}^n\quad \text{if } \Delta x=c\Delta t$$

§ characteristic-curve method

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\{c(x)u(x,t)\}$$

$$c(x) = 1 - \frac{1-c_{\min}}{1+a^2x^2}$$

$$u(x,0) = \exp\left(-\frac{(x-x_0)^2}{b^2}\right)$$

- exact solution:

$$\frac{\partial cu}{\partial t} = c\frac{\partial}{\partial x}\{cu\} \quad \text{or} \quad \frac{\partial Q}{\partial t} = c\frac{\partial Q}{\partial x}, \quad Q \equiv cu$$

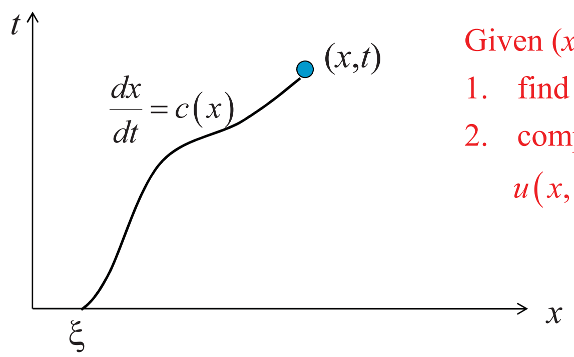
$$\begin{cases} \text{characteristic curve:} & \dfrac{dx}{dt} = c(x) \\ \text{compatibility condition:} & Q(x,t) = c(x)u(x,t) = \text{constant} \end{cases}$$

- characteristic curve : $\dfrac{dx}{dt} = c(x)$

$$t = \int_{\xi}^{x} \frac{dx'}{c(x')} = \text{char. curve passing } (0,\ \xi)$$

$$t = (x-\xi) + \frac{1-c_{\min}}{c_{\min}} \cdot \frac{\sqrt{c_{\min}}}{a}\left\{\tan^{-1}\frac{ax}{\sqrt{c_{\min}}} - \tan^{-1}\frac{a\xi}{\sqrt{c_{\min}}}\right\}$$

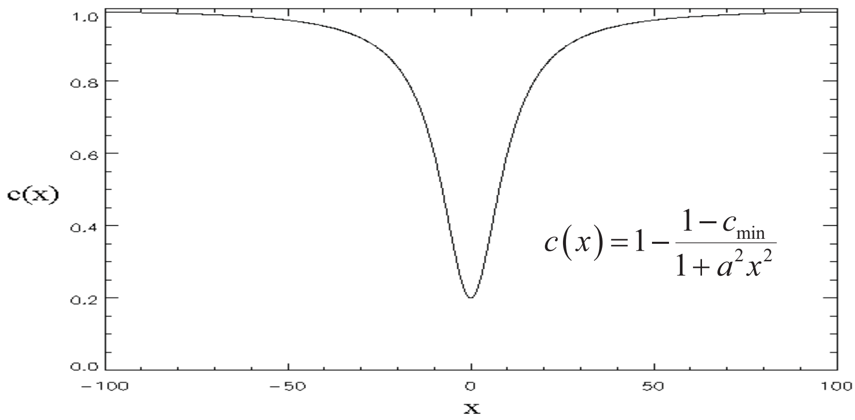
- compatibility condition: $c(x)u(x,t) = c(\xi)u(\xi,0)$



Given (x,t)

1. find ξ
2. compute

   $u(x,t) = c(\xi)u(\xi,0)/c(x)$



$$u(x,0) = \exp\left(-\frac{(x - x_0)^2}{b^2}\right) \qquad x_0 = -50$$