Mean Motion

~ one-point-one-time statistics
~ essential and important but not complete

§ Reynolds (1894) Averaged Velocity

$$u_i = \overline{u}_i + u_i' \text{ = mean + turbulent velocity}$$

mean kinetic energy per unit mass = $\frac{1}{2}\overline{u_i u_i} = \frac{1}{2}\overline{u}_i\overline{u}_i + \frac{1}{2}\overline{u_i' u_i'} \equiv \overline{K} + K$

= energy of mean motion + turbulent energy

(a) Continuity: $\nabla \cdot \vec{u} = \frac{\partial u_j}{\partial x_j} = \frac{\partial(\overline{u}_j + u_j')}{\partial x_j} = 0 \quad (1) \quad \Rightarrow \overline{\frac{\partial u_j}{\partial x_j}} = \frac{\partial \overline{u}_j}{\partial x_j} = 0 \quad (1a)$

$$(1)\text{-}(1a) \Rightarrow \frac{\partial u_j'}{\partial x_j} = 0 \quad (1b)$$

Mean Motion

§ Reynolds (1894) Averaged Navier-Stokes Equations (RANS)

(b) Momentum: $\frac{\partial u_i}{\partial t} + \frac{\partial(u_i u_j)}{\partial x_j} = g_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \tau_{ji}}{\partial x_j} \quad (2)$

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$u_i = \overline{u}_i + u_i' \Rightarrow \overline{u_i u_j} = \overline{(\overline{u}_i + u_i')(\overline{u}_j + u_j')} = \overline{\overline{u}_i\overline{u}_j + \cancel{u_j'\overline{u}_i} + \cancel{u_i'\overline{u}_j} + u_i' u_j'}$$

$$\frac{D\overline{u}_i}{Dt} \equiv \frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j\frac{\partial \overline{u}_i}{\partial x_j} = g_i - \frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\overline{\tau}_{ij} - \rho\overline{u_i' u_j'}\right) \quad (2a)$$

$$\overline{\tau}_{ij} = \mu\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)$$ mean viscous stress tensor

Mean Motion

$$\tau'_{ij} = -\rho\left(\overline{u'_i u'_j} - \frac{1}{3}\delta_{ij}\overline{u'_k u'_k}\right) = \text{Reynolds (turbulent) stress tensor}$$

~ the average momentum flux due to turbulent velocity fluctuations

~ the interaction (coupling) of turbulence with the mean flow

~ arising from the nonlinear (convection) term of Navier-Stokes equations

~ cause the closure problem

~ much larger than viscous stress except near very walls where $\dfrac{\partial \overline{u}_i}{\partial x_j}$ is not small for generally large-Reynolds-number turbulent flows

(As $\mu \to 0$, $\overline{\tau}_{ij} = \mu\left(\dfrac{\partial \overline{u}_i}{\partial x_j} + \dfrac{\partial \overline{u}_j}{\partial x_i}\right) \to 0$ because $\overline{u}_i$ does not fluctuate.)

Mean Motion

(c) Thermal energy equation

$$\overline{\rho c_p\left\{\frac{\partial T}{\partial t} + \frac{\partial}{\partial x_j}(u_j T)\right\} = \frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right)}$$

$u_i = \overline{u}_i + u'_i,\quad T = \overline{T} + T' \quad \Longrightarrow \quad \overline{u_j T} = \overline{\left(\overline{u}_j + u'_j\right)\left(\overline{T} + T'\right)} = \overline{u}_j\overline{T} + \overline{u'_j T'}$

$$\rho c_p\left\{\frac{\partial \overline{T}}{\partial t} + \overline{u}_j\frac{\partial \overline{T}}{\partial x_j}\right\} = \frac{\partial}{\partial x_j}\left(\overset{\text{molecular diffusion}}{k\frac{\partial \overline{T}}{\partial x_j}} - \underset{\text{turbulent convection heat transfer}}{\rho c_p\overline{u'_j T'}}\right)$$

Mean Motion

(d) Turbulent momentum equations: total momentum – mean momentum

$$\frac{Du_i'}{Dt} \equiv \frac{\partial u_i'}{\partial t} + \bar{u}_j \frac{\partial u_i'}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \frac{1}{\rho}\frac{\partial \tau'_{ji}}{\partial x_j} \qquad (2b)$$

$$\tau'_{ij} = \mu\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}\right) + \rho\left(\overline{u_i' u_j'} - \bar{u}_i u'_j - u_i' u'_j\right)$$

(e) Energy of mean motion = $\bar{K} \equiv \bar{u}_i \bar{u}_i / 2$

$$\bar{u}_i \cdot \left\{\frac{D\bar{u}_i}{Dt} = g_i - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{1}{\rho}\frac{\partial \bar{\tau}_{ji}}{\partial x_j}\right\}$$

$$\rho\frac{D\bar{K}}{Dt} = \rho \bar{u}_i g_i - \bar{u}_i \frac{\partial \bar{p}}{\partial x_i} + \bar{u}_i \frac{\partial}{\partial x_j}\mu\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \bar{u}_i \frac{\partial \rho\overline{u_i' u_j'}}{\partial x_j}$$

Mean Motion

Energy of mean motion

pressure work

turbulent transport

$$\rho\frac{D\bar{K}}{Dt} = \rho \bar{u}_i g_i - \frac{\partial\left(\bar{u}_i \bar{p}\right)}{\partial x_i} + \frac{\partial}{\partial x_j}\left\{\mu \bar{u}_i\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) - \bar{u}_i \rho\overline{u_i' u_j'}\right\}$$

body force work

viscous diffusion

$$\boxed{-2\mu \bar{S}_{ij}\bar{S}_{ij}} + \rho\overline{u_i' u_j'}\frac{\partial \bar{u}_i}{\partial x_j}$$

negative mostly

molecular dissipation always negative

turbulent cascade

$$\bar{S}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

Mean Motion

Energy of mean motion (no body force)

$$\rho\frac{D\bar{K}}{Dt} = -2\mu\bar{S}_{ij}\bar{S}_{ij} \quad + \quad \rho\overline{u_i'u_j'}\frac{\partial\bar{u}_i}{\partial x_j}$$

$$\underline{+\frac{\partial}{\partial x_j}\left\{-\bar{u}_j\bar{p} + \mu\bar{u}_i\left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i}\right) - \bar{u}_i\rho\overline{u_i'u_j'}\right\}}$$

~ diffusion due to inhomogeneities

~ vanish when integrate over the whole flow domain

~ vanish in homogeneous turbulence

$$\rho\frac{D\bar{K}}{Dt} = -2\mu\bar{S}_{ij}\bar{S}_{ij} - \left(-\rho\overline{u_i'u_j'}\frac{\partial\bar{u}_i}{\partial x_j}\right)$$

viscous dissipation (irreversible)

energy cascade rate (reversible)

RANS

RANS (Reynolds Averaged Navier-Stokes Equations)

$$\frac{\partial\bar{u}_j}{\partial x_j} = 0$$

$$\frac{D\bar{u}_i}{Dt} \equiv \frac{\partial\bar{u}_i}{\partial t} + \bar{u}_j\frac{\partial\bar{u}_i}{\partial x_j} = g_i - \frac{\partial}{\partial x_i}\left(\frac{\bar{p}}{\rho} + \tfrac{2}{3}K\right) + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\bar{\tau}_{ij} + \tau_{ij}^t\right)$$

(Here overbar represents an ensemble average.)

$$\bar{\tau}_{ij} = \mu\left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i}\right)$$ mean viscous stress tensor

$$\tau_{ij}^t = -\rho\overline{u_i'u_j'} + \tfrac{2}{3}\rho K\delta_{ij} = \text{Reynolds (turbulent) stress tensor}$$

- Zero equation
- One equation Model : model *K*-equation
- Two equation Model : $K-\text{equation} + \varepsilon-\text{equation}$
- Reynolds stress models: model τ_{ij}^t -equations

Turbulent (Eddy) Diffusivities

$$\frac{\tau_{ij}}{\rho} = 2\nu S_{ij}$$

ν : molecular viscosity (Newtonian fluid)

~ fluid property

$$\frac{\tau_{ij}^t}{\rho} = 2\varepsilon_M \bar{S}_{ij}$$

ε_M : momentum eddy viscosity (isotropic form)

~ field property

$$q_j^t = -\rho c_p \overline{u_j' T'} = \text{Reynolds (turbulent) heat flux}$$

$$= \rho c_p \varepsilon_H \frac{\partial \bar{T}}{\partial x_j} \quad \text{(isotropic form)}$$

ε_H : thermal eddy diffusivity

$$Pr_t \equiv \frac{\varepsilon_M}{\varepsilon_H} = \text{turbulent Prandtl number}$$

RANS

§ **Zero equation**

- **Mixing length models**: $\varepsilon_M \approx l^2 \left|\bar{S}\right|$

¶ Prandtl and Karman:

Sublayer: $l \approx y^2$

Overlap layer: $l \approx \kappa y$

Outerlayer: $l \approx \text{constant}$

¶ van Driest Model

$$l \approx \kappa y \underbrace{\left[1 - \exp\left(-\frac{y^+}{A}\right)\right]}_{\text{damping factor}} \; ; \; A \approx 26 \text{ for flat-plate flow}$$

A varies with flow conditions
(pressure gradient, wall roughness,
blowing/suction etc)

RANS

§ One-Equation Model

- **Eddy Viscosity Concept**: $$\frac{\tau^t_{ij}}{\rho} = 2\varepsilon_M \bar{S}_{ij} \quad \text{(divergence-free)}$$

- dimensional analysis:

$$\underset{m^2/s}{\varepsilon_M} = f\left(\underset{m^2/s^2}{K}, \underset{m^2/s^3}{\bar{\varepsilon}}\right) \qquad = C_\mu K^2/\bar{\varepsilon}$$

- turbulent kinetic energy dissipation rate:

$$\bar{\varepsilon} = \frac{\text{drag} \times \text{velocity}}{\text{mass}} \sim \frac{\left|\rho \times \text{velocity}^2 \times \text{area}\right| \times \text{velocity}}{\rho L^3} \sim \frac{K^{3/2}}{L}$$

$K^{1/2} \propto$ eddy velocity
$L \propto$ effective eddy size

$$\bar{\varepsilon} = C_\varepsilon \frac{K^{3/2}}{L}$$

RANS

- turbulent kinetic energy per unit mass *K*:

$$\rho\frac{DK}{Dt} = \rho\frac{\partial K}{\partial t} + \rho\bar{u}_j\frac{\partial K}{\partial x_j}$$

$$= \frac{\partial}{\partial x_j}\left\{-\overline{p'u'_j} - \tfrac{1}{2}\rho\overline{u'_iu'_iu'_j} + \mu\frac{\partial K}{\partial x_j}\right\} - \rho\overline{u'_iu'_j}\frac{\partial \bar{u}_i}{\partial x_j} - \overbrace{2\mu\overline{S'_{ij}S'_{ij}}}^{\bar{\varepsilon}}$$

- turbulent diffusion term:

$$-\frac{1}{\rho}\overline{p'u'_j} - \tfrac{1}{2}\overline{u'_iu'_iu'_j} \cong C_K\left[\frac{\ell^2}{t}\right]\frac{\partial K}{\partial x_j} := C_K\frac{K^2}{\bar{\varepsilon}}\frac{\partial K}{\partial x_j}$$

- turbulent production term:

$$-\rho\overline{u'_iu'_j}\frac{\partial \bar{u}_i}{\partial x_j} = \left(\tau^t_{ij} - \tfrac{2}{3}\rho K\delta_{ij}\right)\frac{\partial \bar{u}_i}{\partial x_j} = \tau^t_{ij}\frac{\partial \bar{u}_i}{\partial x_j}$$

$$\frac{\tau^t_{ij}}{\rho} = 2\varepsilon_M\bar{S}_{ij} = 2C_\mu\frac{K^2}{\bar{\varepsilon}}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

RANS

§ **One-Equation Model**

$$\frac{\partial K}{\partial t}+\bar{u}_j\frac{\partial K}{\partial x_j}=\frac{\partial}{\partial x_j}\left\{C_K\frac{K^2}{\varepsilon}\frac{\partial K}{\partial x_j}+\nu\frac{\partial K}{\partial x_j}\right\}+C_\mu\frac{K^2}{\varepsilon}\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_j}-\bar{\varepsilon}$$

$$\frac{\partial \bar{u}_j}{\partial x_j}=0$$

$$\frac{\partial \bar{u}_i}{\partial t}+\bar{u}_j\frac{\partial \bar{u}_i}{\partial x_j}=g_i-\frac{\partial}{\partial x_i}\left(\frac{\bar{p}}{\rho}+\tfrac{2}{3}K\right)+\frac{\partial}{\partial x_j}\left(\nu\frac{\partial \bar{u}_i}{\partial x_j}+\overbrace{2C_\mu\frac{K^2}{\varepsilon}\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)}^{\tau_{ij}^t/\rho}\right)$$

$$\bar{\varepsilon}=C_\varepsilon\frac{K^{3/2}}{L}$$

6 equations for 6 unknowns $\left(\bar{u}_i\,,\ \frac{\bar{p}}{\rho}+\tfrac{2}{3}K,\ K,\ \bar{\varepsilon}\right)$

with 3 empirical constants $\left(C_K,\ C_\mu,\ C_\varepsilon/L\right)$

RANS

§ **Two-Equation Model**

• turbulent kinetic energy dissipation rate (per unit mass): $\bar{\varepsilon}\equiv\nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}$

exact equation:

$$\frac{D\bar{\varepsilon}}{Dt}=\frac{\partial}{\partial x_j}\left\{\overbrace{-\nu\overline{u'_j\frac{\partial u_i'}{\partial x_m}\frac{\partial u_i'}{\partial x_m}}-\frac{2\nu}{\rho}\overline{\frac{\partial u'_j}{\partial x_m}\frac{\partial p'}{\partial x_m}}}^{\text{turbulent diffusion}}+\underbrace{\nu\frac{\partial \bar{\varepsilon}}{\partial x_j}}_{\text{molecular diffusion}}\right\}$$

$$\overbrace{-2\nu\overline{u'_j\frac{\partial u_i'}{\partial x_m}}\frac{\partial^2\bar{u}_i}{\partial x_j\partial x_m}-2\nu\frac{\partial \bar{u}_i}{\partial x_m}\left\{\overline{\frac{\partial u'_j}{\partial x_i}\frac{\partial u'_j}{\partial x_m}}+\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_m}{\partial x_j}}\right\}}^{\text{production}}$$

$$\overbrace{-2\nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_m}\frac{\partial u'_j}{\partial x_m}}-2\nu\overline{\frac{\partial^2 u_i'}{\partial x_j\partial x_j}\frac{\partial^2 u_i'}{\partial x_m\partial x_m}}}^{\text{destruction}}$$

RANS

- turbulent diffusion terms:

$$-\nu\overline{u'_j\frac{\partial u'_i}{\partial x_m}\frac{\partial u'_i}{\partial x_m}}-\frac{2\nu}{\rho}\overline{\frac{\partial u'_j}{\partial x_m}\frac{\partial p'}{\partial x_m}}\cong\left[\frac{m^2}{s}\right]\frac{\partial\varepsilon}{\partial x_j}:\equiv C_\varepsilon\frac{K^2}{\varepsilon}\frac{\partial\varepsilon}{\partial x_j}$$

- production terms:

$$-2\nu\overline{u'_j\frac{\partial u'_i}{\partial x_m}}\frac{\partial^2\bar{u}_i}{\partial x_j\partial x_m}-2\nu\frac{\partial\bar{u}_i}{\partial x_m}\left\{\overline{\frac{\partial u'_j}{\partial x_i}\frac{\partial u'_j}{\partial x_m}}+\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_m}{\partial x_j}}\right\}$$

$$\cong\left[\frac{m^3}{kg\cdot s}\right]\cdot\tau^t_{ij}\frac{\partial\bar{u}_i}{\partial x_j}\quad:\equiv -C_{\varepsilon 1}\frac{\varepsilon}{K}\overline{u'_i u'_j}\frac{\partial\bar{u}_i}{\partial x_j}$$

$$:\equiv 2C_\mu C_{\varepsilon 1}K\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)\frac{\partial\bar{u}_i}{\partial x_j}$$

- destruction terms:

$$-2\nu\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_i}{\partial x_m}\frac{\partial u'_j}{\partial x_m}}-2\nu\overline{\frac{\partial^2 u'_i}{\partial x_j\partial x_j}\frac{\partial^2 u'_i}{\partial x_m\partial x_m}}\cong\left[\frac{1}{\mathrm{sec}}\right]\bar{\varepsilon}\quad:\equiv -C_{\varepsilon 2}\frac{\bar{\varepsilon}}{K}\cdot\bar{\varepsilon}$$

RANS

§ **Two-Equation Model**

$$\frac{\partial K}{\partial t}+\bar{u}_j\frac{\partial K}{\partial x_j}=\frac{\partial}{\partial x_j}\left\{C_K\frac{K^2}{\varepsilon}\frac{\partial K}{\partial x_j}+\nu\frac{\partial K}{\partial x_j}\right\}+C_\mu\frac{K^2}{\varepsilon}\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)\frac{\partial\bar{u}_i}{\partial x_j}-\bar{\varepsilon}$$

$$\frac{\partial\bar{u}_j}{\partial x_j}=0$$

$$\frac{\partial\bar{u}_i}{\partial t}+\bar{u}_j\frac{\partial\bar{u}_i}{\partial x_j}=g_i-\frac{\partial}{\partial x_i}\left(\frac{\bar{p}}{\rho}+\frac{2}{3}K\right)+\frac{\partial}{\partial x_j}\left(\nu\frac{\partial\bar{u}_i}{\partial x_j}+2C_\mu\frac{K^2}{\varepsilon}\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)\right)$$

$$\frac{D\bar{\varepsilon}}{Dt}=\frac{\partial}{\partial x_l}\left\{C_\varepsilon\frac{K^2}{\varepsilon}\frac{\partial\varepsilon}{\partial x_l}+\nu\frac{\partial\varepsilon}{\partial x_l}\right\}+2C_\mu C_{\varepsilon 1}K\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)\frac{\partial\bar{u}_i}{\partial x_j}-C_{\varepsilon 2}\frac{\bar{\varepsilon}^2}{K}$$

6 equations for 6 unknowns $\left(\bar{u}_i\,,\ \frac{\bar{p}}{\rho}+\frac{2}{3}K,\ K,\ \bar{\varepsilon}\right)$

with 5 empirical constants $\left(C_K,\ C_\mu,\ C_\varepsilon, C_{\varepsilon 1}, C_{\varepsilon 2}\right)$

RANS

§ Reynolds-stress Model

RANS (Reynolds Averaged Navier-Stokes Equations)

$$\frac{\partial \bar{u}_j}{\partial x_j} = 0$$

$$\frac{D\bar{u}_i}{Dt} \equiv \frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = g_i - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\bar{\tau}_{ij} + \tau^t_{ij}\right)$$

$$\bar{\tau}_{ij} = \mu\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$ mean viscous stress tensor

$$\tau^t_{ij} = -\rho\,\overline{u'_i u'_j} = \text{Reynolds (turbulent) stress tensor}$$

Model τ^t_{ij} equations directly.

RANS

Reynolds stress tensor equations

$$\frac{D\overline{u'_i u'_j}}{Dt} = \frac{\partial \overline{u'_i u'_j}}{\partial t} + \bar{u}_m \frac{\partial \overline{u'_i u'_j}}{\partial x_m}$$ ~ mean motion Lagrangian

~ pressure effects (nonlocal, linear, and nonlinear)

$$= -\frac{\partial}{\partial x_m}\left\{\overline{\left(u'_j \delta_{im} + u'_i \delta_{jm}\right)\frac{p'}{\rho}}\right\} + \overline{\frac{p'}{\rho}\left(\frac{\partial u'_j}{\partial x_i} + \frac{\partial u'_i}{\partial x_j}\right)}$$

$$+\nu\left(\frac{\partial^2 \overline{u'_i u'_j}}{\partial x_m \partial x_m} - 2\overline{\frac{\partial u'_i}{\partial x_m}\frac{\partial u'_j}{\partial x_m}}\right)$$ ~ viscous diffusion/dissipation effect

nonlocal

$$-\left(\overline{u'_i u'_m}\frac{\partial \bar{u}_j}{\partial x_m} + \overline{u'_j u'_m}\frac{\partial \bar{u}_i}{\partial x_m}\right)$$ ~ production and reorientation by the mean motion

$$-\frac{\partial \overline{u'_i u'_j u'_m}}{\partial x_m}$$ ~ turbulent advection

RANS

- turbulent diffusion terms:

$$-\overline{\left(u'_j\delta_{im}+u'_i\delta_{jm}\right)\frac{p'}{\rho}}-\overline{u'_iu'_ju'_m} \cong \left[\frac{m^2}{\sec}\right]\frac{\partial\overline{u'_iu'_j}}{\partial x_m} :\equiv C_K\frac{K^2}{\varepsilon}\cdot\frac{\partial\overline{u'_iu'_j}}{\partial x_m}$$

- pressure-strain terms:

$$\overline{\frac{p'}{\rho}\left(\frac{\partial u'_j}{\partial x_i}+\frac{\partial u'_i}{\partial x_j}\right)}$$

traceless, expected to be able to be expressed in terms of $\frac{\partial\overline{u}_i}{\partial x_j}$ and $-\overline{u'_iu'_j}$

$$\cong\left[-\overline{u'_iu'_j}\right]\left[\frac{\partial\overline{u}_i}{\partial x_j}\right] :\equiv C_2\left\{\overline{u'_iu'_m}\frac{\partial\overline{u}_j}{\partial x_m}+\overline{u'_ju'_m}\frac{\partial\overline{u}_i}{\partial x_m}-\frac{2}{3}\delta_{ij}\overline{u'_nu'_m}\frac{\partial\overline{u}_n}{\partial x_m}\right\}$$

- dissipation terms:

non-isotropic part

$$-2\nu\overline{\frac{\partial u'_i}{\partial x_m}\frac{\partial u'_j}{\partial x_m}} :\equiv -\frac{2}{3}\delta_{ij}\varepsilon-C_1\frac{\varepsilon}{K}\left(\overline{u'_iu'_j}-\frac{2}{3}\delta_{ij}K\right)$$

isotropic part

RANS

modeled Reynolds stress tensor equations

$$\frac{D\overline{u'_iu'_j}}{Dt}=\frac{\partial}{\partial x_m}\left\{\left(C_K\frac{K^2}{\varepsilon}+\nu\right)\frac{\partial\overline{u'_iu'_j}}{\partial x_m}\right\}-\left\{\overline{u'_iu'_m}\frac{\partial\overline{u}_j}{\partial x_m}+\overline{u'_ju'_m}\frac{\partial\overline{u}_i}{\partial x_m}\right\}$$
$$+C_2\left\{\overline{u'_iu'_m}\frac{\partial\overline{u}_j}{\partial x_m}+\overline{u'_ju'_m}\frac{\partial\overline{u}_i}{\partial x_m}-\frac{2}{3}\delta_{ij}\overline{u'_nu'_m}\frac{\partial\overline{u}_n}{\partial x_m}\right\}$$
$$-\frac{2}{3}\delta_{ij}\overline{\varepsilon}-C_1\frac{\overline{\varepsilon}}{K}\left(\overline{u'_iu'_j}-\frac{2}{3}\delta_{ij}K\right)$$

6 equations for 6 new unknowns $\left(\overline{u_1'^2},\ \overline{u_2'^2},\ \overline{u_3'^2},\ \overline{u'_1u'_2},\ \overline{u'_2u'_3},\ \overline{u'_3u'_1}\right)$

i=j :

$$\frac{DK}{Dt}=\frac{\partial}{\partial x_m}\left\{\left(C_K\frac{K^2}{\varepsilon}+\nu\right)\frac{\partial K}{\partial x_m}\right\}-\frac{1}{2}\left\{\overline{u'_iu'_m}\frac{\partial\overline{u}_i}{\partial x_m}+\overline{u'_iu'_m}\frac{\partial\overline{u}_i}{\partial x_m}\right\}-\overline{\varepsilon}$$

RANS

§ **Reynolds-stress Model**

$$\frac{D\bar{\varepsilon}}{Dt}=\frac{\partial}{\partial x_l}\left\{C_\varepsilon\frac{K^2}{\varepsilon}\frac{\partial\varepsilon}{\partial x_l}+\nu\frac{\partial\varepsilon}{\partial x_l}\right\}-C_{\varepsilon 1}\frac{\varepsilon}{K}\overline{u'_i u'_j}\frac{\partial\bar{u}_i}{\partial x_j}-C_{\varepsilon 2}\frac{\varepsilon^2}{K}$$

11 equations for 11 unknowns

$$\left(\bar{u}_1,\bar{u}_2,\bar{u}_3,\frac{\bar{p}}{\rho},\bar{\varepsilon},\overline{u_1'^2},\ \overline{u_2'^2},\ \overline{u_3'^2},\ \overline{u'_1u'_2},\ \overline{u'_2u'_3},\ \overline{u'_3u'_1}\right)$$

$$\frac{\partial\bar{u}_j}{\partial x_j}=0$$

$$\frac{\partial\bar{u}_i}{\partial t}+\bar{u}_j\frac{\partial\bar{u}_i}{\partial x_j}=g_i-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x_i}+\frac{\partial}{\partial x_j}\left(\nu\frac{\partial\bar{u}_i}{\partial x_j}-\overline{u'_iu'_j}\right)$$

$$\frac{D\overline{u'_iu'_j}}{Dt}=\frac{\partial}{\partial x_m}\left\{\left(C_K\frac{K^2}{\varepsilon}+\nu\right)\frac{\partial\overline{u'_iu'_j}}{\partial x_m}\right\}-\left\{\overline{u'_iu'_m}\frac{\partial\bar{u}_j}{\partial x_m}+\overline{u'_ju'_m}\frac{\partial\bar{u}_i}{\partial x_m}\right\}$$
$$+C_2\left\{\overline{u'_iu'_m}\frac{\partial\bar{u}_j}{\partial x_m}+\overline{u'_ju'_m}\frac{\partial\bar{u}_i}{\partial x_m}-\frac{2}{3}\delta_{ij}\overline{u'_nu'_m}\frac{\partial\bar{u}_n}{\partial x_m}\right\}$$
$$-\frac{2}{3}\delta_{ij}\bar{\varepsilon}-C_1\frac{\varepsilon}{K}\left(\overline{u'_iu'_j}-\frac{2}{3}\delta_{ij}K\right)$$

RANS

§ **Algebraic Stress Model**

Assume negligible turbulent convection and diffusion

$$\cancel{\frac{D\overline{u'_iu'_j}}{Dt}}=\cancel{\frac{\partial}{\partial x_m}\left\{\left(C_K\frac{K^2}{\varepsilon}+\nu\right)\frac{\partial\overline{u'_iu'_j}}{\partial x_m}\right\}}-\left\{\overline{u'_iu'_m}\frac{\partial\bar{u}_j}{\partial x_m}+\overline{u'_ju'_m}\frac{\partial\bar{u}_i}{\partial x_m}\right\}$$
$$+C_2\left\{\overline{u'_iu'_m}\frac{\partial\bar{u}_j}{\partial x_m}+\overline{u'_ju'_m}\frac{\partial\bar{u}_i}{\partial x_m}-\frac{2}{3}\delta_{ij}\overline{u'_nu'_m}\frac{\partial\bar{u}_n}{\partial x_m}\right\}$$
$$-\frac{2}{3}\delta_{ij}\bar{\varepsilon}-C_1\frac{\varepsilon}{K}\left(\overline{u'_iu'_j}-\frac{2}{3}\delta_{ij}K\right)$$

$$0=-\left\{\overline{u'_iu'_m}\frac{\partial\bar{u}_j}{\partial x_m}+\overline{u'_ju'_m}\frac{\partial\bar{u}_i}{\partial x_m}\right\}-\frac{2}{3}\delta_{ij}\bar{\varepsilon}-C_1\frac{\varepsilon}{K}\left(\overline{u'_iu'_j}-\frac{2}{3}\delta_{ij}K\right)$$
$$+C_2\left\{\overline{u'_iu'_m}\frac{\partial\bar{u}_j}{\partial x_m}+\overline{u'_ju'_m}\frac{\partial\bar{u}_i}{\partial x_m}-\frac{2}{3}\delta_{ij}\overline{u'_nu'_m}\frac{\partial\bar{u}_n}{\partial x_m}\right\}$$

~ algebraic equations for the Reynolds stresses ~

Turbulent Prandtl Number Pr_t

Kays, ASME J Heat Transfer 116, 284-295(1994)

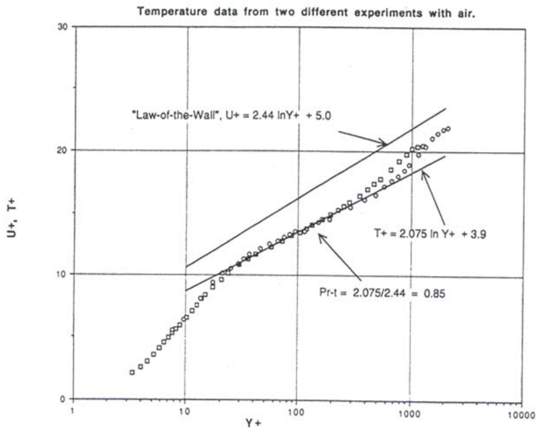

Fig. 1 Example of determination of turbulent Prandtl number from slope of T^+ curve in the "log" region, data of Blackwell et al. (1972)

No pressure gradient
No blowing/suction
No surface curvature

In log-region:

$$u^+ = 2.44 \ln y^+ + 5.00$$

$$T^+ = C_1 \ln y^+ + C_2$$

$$-\rho \overline{u'v'} \approx \tau_0$$

$$-\rho c_p \overline{v'T'} \approx q_0''$$

$$\mathrm{Pr}_t \equiv \frac{\varepsilon_M}{\varepsilon_H} = \frac{\overline{u'v'} \cdot \partial \overline{T} / \partial y}{\overline{v'T'} \cdot \partial \overline{u} / \partial y} = \frac{C_1}{2.44}$$

Turbulent Prandtl Number Pr_t

Yakhot et al (1987): Renormalization Group Method

$$\mathrm{Pr}_{eff} \equiv \frac{1 + \varepsilon_M / \nu}{\dfrac{\varepsilon_M / \nu}{\mathrm{Pr}_t} + \dfrac{1}{\mathrm{Pr}}}$$

$$\frac{1}{1 + \varepsilon_M / \nu} = \left\{ \frac{\left(1/\mathrm{Pr}_{eff} - 1.1793\right)}{\left(1/\mathrm{Pr} - 1.1793\right)} \right\}^{0.65} \left\{ \frac{\left(1/\mathrm{Pr}_{eff} + 2.1793\right)}{\left(1/\mathrm{Pr} + 2.1793\right)} \right\}^{0.35}$$

$$\mathrm{Pr}_t \to 0.85 \text{ as } Pe_t \equiv \mathrm{Pr} \cdot \frac{\varepsilon_M}{\nu} = \frac{\varepsilon_M}{\alpha} \to \infty$$

$\mathrm{Pr}_t \uparrow\uparrow$ at small Pe_t

fitting curve: $\mathrm{Pr}_t = 0.7 / Pe_t + 0.85$

Turbulent Prandtl Number Pr_t

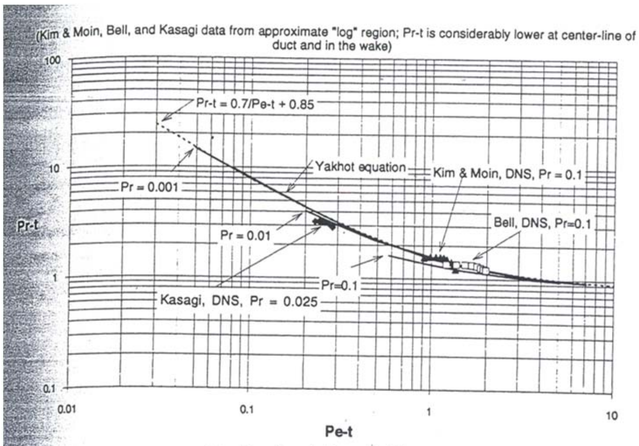

Fig. 2 Analytic solutions

Yakhot vs DNS data (low Reynolds numbers)

Turbulent Prandtl Number Pr_t

Two branches at low Pe_t:

$$\begin{cases} \mathrm{Pr}_t = 0.7/Pe_t + 0.85 \\ \mathrm{Pr}_t = 2.0/Pe_t + 0.85 \end{cases}$$

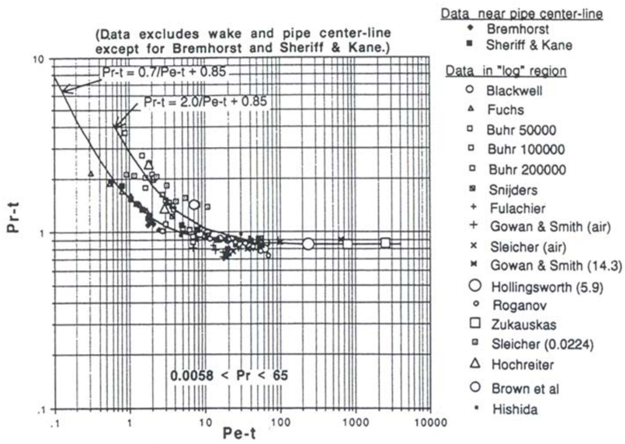

Fig. 4 Turbulent Prandtl number in the "logarithmic" region, $0.0058 < \mathrm{Pr} < 65$

(experimental data of fully developed pipe flows and external flat-plate boundary layers)

Turbulent Prandtl Number Pr_t

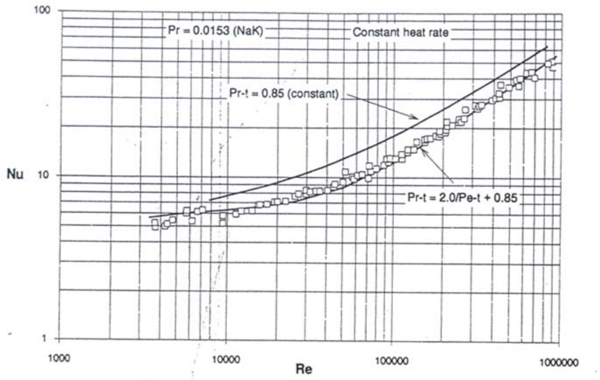

Fig. 5 Skupinski et al., experiments

fully-developed flow in circular pipes with constant wall heat flux
mixing-length eddy diffusivity model
$\Pr_t = 2.0/Pe_t + 0.85$ used across the entire pipe

Nu decreases with increasing $\Pr_t$.

Turbulent Prandtl Number Pr_t

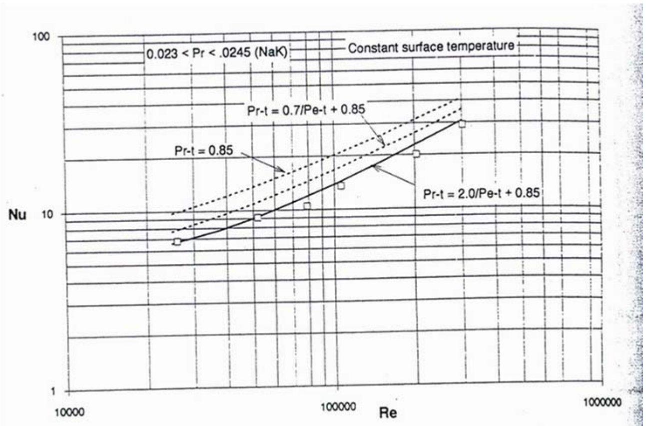

Fig. 7 Sleicher et al., experiments

fully developed flow in circular pipes with constant wall temperature

thermal boundary condition effect?

Turbulent Prandtl Number

W. M. Kays, "Turbulent Prandtl number—where are we," ASME J. Heat Transfer 116, 284 1994.

Constant model: $Pr_t = 0.9$

Kays-Crawford model:

$$Pr_t = \left\{ \frac{1}{2\,Pr_{t\infty}} + CPe_t\sqrt{\frac{1}{Pr_{t\infty}}} - \left(CPe_t\right)^2\left[1 - exp\left(-\frac{1}{CPe_t\sqrt{Pr_{t\infty}}}\right)\right]\right\}$$

$$Pe_t \equiv Pr\cdot\frac{\varepsilon_M}{\nu}$$

$$C = 0.3$$

$$Pr_{t\infty} = 0.86 \begin{cases}\text{for gases and light liquids,}\\ \text{and for Pr<0.6 (liquid metals)}\end{cases}$$

Weigand correction:

$$Pr_{t\infty} = 0.85 + \frac{100}{Pr\,Re^{0.888}}$$

This model has been calibrated for equilibrium turbulent boundary layers for use with the mixing-length turbulence model. It works equally well for turbulent flows with one- and 2-equation turbulence models. This option is not recommended for transitional boundary layer flows or flows with pressure gradients, or for liquids with Pr > 10-20.

Turbulent Prandtl Number Pr_t

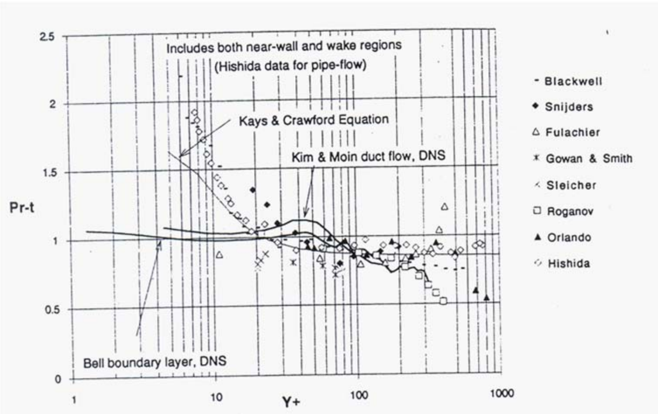

Fig. 9 Pr-t data for air

A marked increase in Pr_t for $y^+<30$!

a phenomenon not seen from the DNS data

Hishida: pipe flow
Others: external boundary flat-plate layers

Turbulent Prandtl Number Pr_t

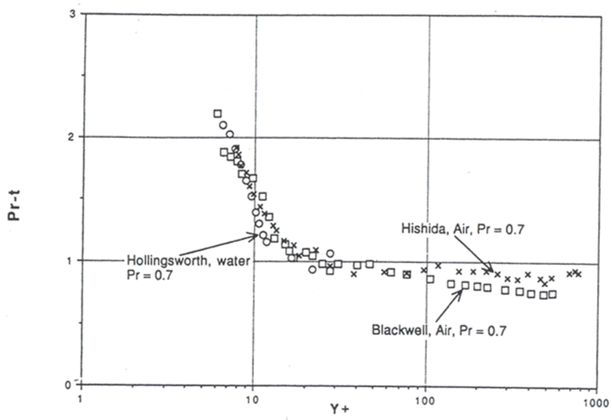

Fig. 10 Comparison of data for air and water

The data for water are almost identical to those for air!

Turbulent Prandtl Number Pr_t

- large Pr

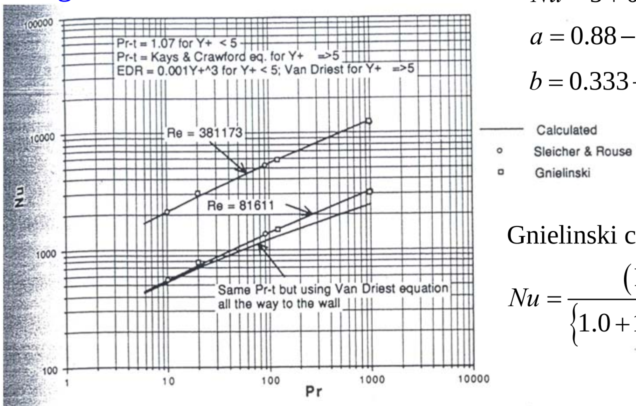

Fig. 19 Fully developed flow in a circular tube

Sleicher and Rouse correlation:

$$Nu = 5 + 0.015\,\mathrm{Re}^a\,\mathrm{Pr}^b$$

$$a = 0.88 - 0.24/(4 + \mathrm{Pr})$$

$$b = 0.333 + 0.5\exp(-0.6\,\mathrm{Pr})$$

Gnielinski correlation:

$$Nu = \frac{(\mathrm{Re} - 1000)\,\mathrm{Pr}\cdot c_f/2}{\left\{1.0 + 12.7\sqrt{c_f/2}\left(\mathrm{Pr}^{2/3} - 1.0\right)\right\}}$$

- As Pr increases, the temperature profile moves closer and closer to the wall but still $\varepsilon_H >> \alpha$.

- Pr_t becomes lower and approaches 1.00 at the wall.

Turbulent Prandtl Number Pr_t

Remark:

- Pr_t appears to be primarily a function of a turbulent Peclet number Pe_t.
- Pr_t approaches to a constant value of about 0.85 at very large Pe_t.
- At small values of Pe_t, Pr_t increases indefinitely.
- Pr_t becomes lower and approaches 1.00 at the wall.
- A use of the "log" region Pr_t is usually sufficiently accurate.
- There may be a pressure gradient effect on Pr_t.
- Blowing/suction has little effect upon Pr_t.
- Surface roughness has little effect upon Pr_t.