9. Kolmogorov's Theories

1. C.R. Acad. Sci. USSR 30 (1941) 301-305
2. C.R. Acad. Sci. USSR 30 (1941) 538
3. C.R. Acad Sci. USSR 32 (1941) 16-18

L : length scale of energy-containing eddies

η: length scale of dissipative eddies

q : characteristic turbulent velocity

ε : energy dissipation rate

Assumptions:

- sufficiently large Reynolds number : $L/\eta \sim \mathrm{Re}_L^{3/4} >> 1$
- no intermittency in ε : $\varepsilon(\vec{x}) \approx \bar{\varepsilon}$
- localness

$$\mathrm{Re}_L \equiv \frac{qL}{\nu}$$

9. Kolmogorov's Theories

Theory1: There exist eddies (called equilibrium eddies) which motion is not affected by large-scale effects provided $\mathrm{Re}_L^{3/4} >> 1$

Consequence:

These eddies are dominated statistically by parameter ε and ν only.

Belief:

Given ε, the flow will adjust itself so that the smallest length scale

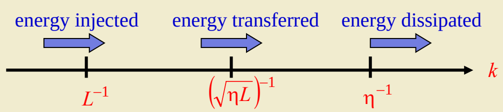
$$\eta \sim \left(\nu^3/\varepsilon\right)^{1/4}$$

and the energy dissipation rate is just as much as that transferred from large eddies.

9. Kolmogorov's Theories

Theory2: There exist eddies of intermediate sizes (called inertial eddies) which motion is not affected by large-scale effects and suffers negligible dissipation, provided $\mathrm{Re}_L^{3/8} >> 1$

$$\eta << \text{ inertial length scale} \sim \sqrt{\eta L} << L$$



Consequence:

The motion of inertial eddies are dominated only by ε.

$$T(k) = 0 \quad \text{and} \quad \int_k^{\infty} T(k)dk = \varepsilon$$

9. Kolmogorov's Theories

Theory3: (Localness) The motion of inertial eddies of scale $r \sim k^{-1}$, if exist, is determined statistically by ε and k alone.

In wave space: $E(k) = f(\varepsilon, k)$

Dimensional analysis: $E(k) = C_K \varepsilon^{2/3} k^{-5/3}$

~ Kolmogorov spectrum ~

In physical space: $\varepsilon \sim \upsilon_r^2 / (r/\upsilon_r)$

υ_r ~ characteristic velocity of eddies of scale r

$$\upsilon_r^2 \sim (\varepsilon r)^{2/3} \sim kE(k)$$

2nd order velocity structure function:

$$F_2(r,t) \equiv \left\langle \left| \vec{u}(\vec{x},t) - \vec{u}(\vec{x}+\vec{r},t) \right|^2 \right\rangle \sim \upsilon_r^2 \sim (\varepsilon r)^{2/3}$$

9. Kolmogorov's Spectrum

Self-similar decay:

There exist some velocity $U(t)$ and some length scale $\Im(t)$ such that

$$E(k,t) = U^2(t)\Im(t)F(k\Im) \approx U^2(t)\Im(t)\cdot(k\Im)^n$$

(1) energy-containing eddies:

$$U \sim q \quad \text{and} \quad \Im \sim L \sim \frac{q^3}{\varepsilon}$$

$$E(k,t) \sim q^2 k^n \left(\frac{q^3}{\varepsilon}\right)^{n+1}$$

Tend to be independent of q as scales $\to$ inertial subrange

$$2 + 3(n+1) = 0 \quad \Rightarrow \quad n = -5/3$$

$$E(k,t) \sim \varepsilon^{2/3} k^{-5/3}$$

9. Kolmogorov's Spectrum

Self-similar decay:

There exist some velocity $U(t)$ and some length scale $\Im(t)$ such that

$$E(k,t) = U^2(t)\Im(t)F(k\Im) \approx U^2(t)\Im(t)\cdot(k\Im)^n$$

(2) dissipative eddies:

$$U \sim (\varepsilon\nu)^{1/4} \quad \text{and} \quad \Im \sim \eta \sim \left(\nu^3/\varepsilon\right)^{1/4}$$

$$E(k,t) \sim (\varepsilon\nu)^{1/2} k^n \left(\nu^3/\varepsilon\right)^{(n+1)/4}$$

Tend to be independent of ν as scales $\to$ inertial subrange

$$\frac{1}{2} + \frac{3(n+1)}{4} = 0 \quad \Rightarrow \quad n = -5/3$$

$$E(k,t) \sim \varepsilon^{2/3} k^{-5/3}$$

9. Kolmogorov's Spectrum

Phenomenological models

- vortex stretching
- vortex interaction
- statistics (random distribution/orientation)

•Tennekes (1968, Phys. Fluids 11,669)

vortex-tube model : $E(k) \sim k^{-1}$

• Corrsin (1962, Phys. Fluids 5, 1301)

vortex-sheet model : $E(k) \sim k^{-2}$

• Lundgren (1982, Phys Fluids 25, 2193)

strained spiral vortex model : $E(k) \sim k^{-5/3}$

9. Intermittency Models

$\varepsilon(\vec{x})$ not uniformly distributed in space/time (intermittent)

• Kolmogorov (1962, JFM13, 82)

Assume $\varepsilon(\vec{x})$ has a log-normal distribution.

Probability distribution function of $\log(\varepsilon/\bar{\varepsilon})$ is Gaussian.

$$\xi \equiv \log(\varepsilon/\bar{\varepsilon}) \quad , \quad P(\xi) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(\xi-\bar{\xi})}{2\sigma^2}\right)$$

Because $P(\xi)d\xi = P(\varepsilon)d\varepsilon$

$$P(\varepsilon) = \frac{1}{\sqrt{2\pi}\sigma\varepsilon}\exp\left(-\frac{\left(\log(\varepsilon/\bar{\varepsilon}) - \overline{\log(\varepsilon/\bar{\varepsilon})}\right)}{2\sigma^2}\right)$$

$$\bar{\varepsilon} = \int_0^\infty P(\varepsilon)d\varepsilon \quad \Rightarrow \quad \bar{\xi} = \overline{\log(\varepsilon/\bar{\varepsilon})} = -\frac{\sigma^2}{2}$$

9. Intermittency Models

$$\overline{\varepsilon^p} = \int_0^\infty \varepsilon^p P(\varepsilon) d\varepsilon = \bar{\varepsilon}^p \exp\left(\frac{\sigma^2}{2} p(p-1)\right)$$

In particular, $p = 2$: $\overline{\varepsilon^2} = \bar{\varepsilon}^2 e^{\sigma^2} \sim \bar{\varepsilon}^2 \left(\frac{L}{r}\right)^{\mu}$ (assumed)

$$\Rightarrow \quad \overline{\varepsilon^p} = \bar{\varepsilon}^p \left(\frac{L}{r}\right)^{\frac{1}{2}\mu p(p-1)} \quad ; \quad \overline{\varepsilon^{2/3}} = \bar{\varepsilon}^{2/3} \left(\frac{L}{r}\right)^{-\frac{1}{9}\mu}$$

Recall $\upsilon_r^2 \sim (\varepsilon r)^{2/3} \sim kE(k)$ (no intermittency)

$\overline{\upsilon_r^2} \sim \overline{(\varepsilon r)^{2/3}} \sim kE(k)$ (corrected)

$$\Rightarrow \quad E(k) \sim \bar{\varepsilon}^{2/3} k^{-\frac{3}{5}} (kL)^{-\frac{1}{9}\mu}$$ (experiments: $\mu \approx 0.5$)

9. Intermittency Models

- Gurvich and Yaglom (1967, Phys. Fluids Supp, S59)

❖ 假想一尺度為 ξ_0 的漩渦，因不穩定 “碎成” C (>>1) 個尺度為ξ_1 的漩渦，則 ，並假設這 C 個漩渦中只有 M (<<C) 個是活躍的 (劇烈區)，亦即能量只傳至這 M 個漩渦中。

$$\xi_1 \sim \xi_o C^{-1/3}$$

❖ M 個尺度為ξ_1 的漩渦，每一個都將因不穩定再 “碎成” C 個尺度為ξ_2的漩渦，，其中又只有 M (<<C) 個是活躍的。

$$\xi_2 \sim \xi_1 C^{-1/3} \sim \xi_o C^{-2/3}$$

❖ 在第 n 階時，尺度為 $\xi_n \sim \xi_o C^{-n/3}$，漩渦總數為 C^n 個，其中只有個是活躍的 M^n。

$$D \equiv \underset{n\to\infty}{\text{limit}}\left(-\frac{\log(\text{活躍漩渦數})}{\log(\text{尺度})}\right) = \underset{n\to\infty}{\text{limit}}\left(-\frac{\log(M^n)}{\log(\xi_o C^{-n/3})}\right) = \frac{3\log(M)}{\log(C)}$$

9. Intermittency Models

$$r \sim \xi_p \sim \xi_o C^{-p/3}$$

$\bar{\varepsilon}_r \equiv$ averaged dissipation rate over active eddies of size r

$$\bar{\varepsilon} \sim \bar{\varepsilon}_r \left(\frac{M}{C}\right)^p$$

$$\overline{\varepsilon^n} \sim \bar{\varepsilon}_r^n \left(\frac{M}{C}\right)^p \sim \bar{\varepsilon}^n \left(\frac{M}{C}\right)^{-p(n-1)} \sim \bar{\varepsilon}^n \left(\frac{\xi_0}{r}\right)^{\mu(n-1)}$$

$$\overline{\varepsilon^2} \sim \bar{\varepsilon}^2 \left(\frac{M}{C}\right)^{-p} \sim \bar{\varepsilon}^2 \left(\frac{\xi_0}{r}\right)^{\mu} \quad (\mu = 3 - D)$$

$n = 2/3$

$$\Rightarrow \quad E(k) \sim \bar{\varepsilon}^{2/3} k^{-\frac{3}{5}} (kL)^{-\frac{1}{3}\mu}$$

9. Intermittency Models

- Gurvich and Yaglom (1967, Phys. Fluids Supp, S59)
  log-normal model
- Frisch et al (1978, JFM 87, 719)
  β model
- She et al (1991, Proc. R. Soc. Lond. A 434, 101)
  (1991, Phys. Rev. Lett. 66, 600)
  two-fluid model
- Meneveau and Sreenivasan (1987, Phys. Rev. Lett. **59**, 1424)
  multi-fractal model
- Benzi et al (1984, J. Phys. A **17**, 3521)
  random β model

9. Spectral theories and stochastic models

Closure hypothesis: an arbitrary relation between velocity moments

Basic assumption: Turbulence is close to Gaussianity.

Experimental evidence: Turbulent velocity p.d.f. is close to a Gaussian but the velocity derivative p.d.f. is closer to an exponential.

In fact, it cannot be a Gaussian; otherwise, there would be no energy exchange between wave modes because of the zero third-order moment.

Models:

Quasi-Normal approximation (**QN**)

Eddy-Damped Quasi-Normal approximation (**EDQN**)

Eddy-Damped Quasi-Normal Markovian approximation (**EDQNM**)

Direct Interaction Approximation (**DIA**)

9. Spectral theories and stochastic models

Gaussian Random Function:

Let $X \equiv (\vec{x}, t)$ be a 4-dimensional variable and

$g(X)$ be a random function of X of zero mean.

Definition: $g(X)$ is a Gaussian random function if given N arbitrary numbers α_i and N values X_i of X, the linear combination $\sum_{i=1}^{N} \alpha_i g(X_i)$ is a Gaussian random variable.

Properties: (i) For any X, $g(X)$ is a Gaussian random variable.

(ii) The odd moments of $g(X)$ are zero.

(iii) The even moments of $g(X)$ can be expressed in terms of 2nd order moments.

$$< g_1 g_2 g_3 g_4 > = < g(X_1) g(X_2) g(X_3) g(X_4) >$$
$$= < g_1 g_2 >< g_3 g_4 > + < g_1 g_3 >< g_2 g_4 > + < g_1 g_4 >< g_2 g_3 >$$

9. Spectral theories and stochastic models

Recall $\frac{\partial \hat{u}_i}{\partial t} + i\Delta_{ij} k_m \left(\hat{u}_j \otimes \hat{u}_m\right) = -\nu k^2 \hat{u}_i \quad ; \quad \Delta_{ij}\left(\vec{k}\right) \equiv \delta_{ij} - \frac{k_i k_j}{k^2}$

(9.1) $$\left(\frac{\partial}{\partial t} + \nu k^2\right)\hat{u}_i = -i\Delta_{ij} k_m \iint_{\vec{p}+\vec{q}-\vec{k}=0} \hat{u}_j(\vec{p})\hat{u}_m(\vec{q})d\vec{p}d\vec{q}$$

(9.2) $$\left(\frac{\partial}{\partial t} + 2\nu k^2\right)\Phi_{ij} = T_{ij}$$

$$\overline{\hat{u}_i(\vec{k},t)\hat{u}_j(\vec{k}',t)} = \Phi_{ij}(\vec{k},t)\delta\left(\vec{k}+\vec{k}'\right)$$

$$T_{ij} = ik_m\left(\Delta_{jn}\Theta_{inm}(\vec{k}) - \Delta_{in}\Theta_{jnm}(-\vec{k})\right)$$

$$\Theta_{inm}(\vec{k}) \equiv \iint_{\vec{p}+\vec{q}+\vec{k}=0} \overline{\hat{u}_i(\vec{k})\hat{u}_n(\vec{p})\hat{u}_m(\vec{q})}d\vec{p}d\vec{q}$$

9. Spectral theories and stochastic models

(9.3) $$\left(\frac{\partial}{\partial t} + \nu\left(k^2 + p^2 + q^2\right)\right) < \hat{u}_i\left(\vec{k}\right)\hat{u}_j(\vec{p})\hat{u}_\ell(\vec{q}) > = R_{ij\ell}\left(\vec{k},\vec{p},\vec{q}\right)$$

$$R_{ij\ell}(\vec{k}) \equiv -ik_m\Delta_{in}(\vec{k}) \iint_{\vec{p}'+\vec{q}'-\vec{k}=0} \overline{\hat{u}_n(\vec{p}')\hat{u}_m(\vec{q}')\hat{u}_j(\vec{p})\hat{u}_\ell(\vec{q})}\; d\vec{p}'d\vec{q}'$$
$$-ip_m\Delta_{jn}(\vec{p}) \iint_{\vec{p}'+\vec{q}'-\vec{p}=0} \overline{\hat{u}_n(\vec{p}')\hat{u}_m(\vec{q}')\hat{u}_i(\vec{k})\hat{u}_\ell(\vec{q})}\; d\vec{p}'d\vec{q}'$$
$$-iq_m\Delta_{\ell n}(\vec{q}) \iint_{\vec{p}'+\vec{q}'-\vec{q}=0} \overline{\hat{u}_n(\vec{p}')\hat{u}_m(\vec{q}')\hat{u}_i(\vec{k})\hat{u}_j(\vec{p})}\; d\vec{p}'d\vec{q}'$$

$$< \hat{u}_i\left(\vec{k}\right)\hat{u}_j(\vec{p})\hat{u}_\ell(\vec{q}) > = \int_0^t R_{ij\ell}\left(\vec{k},\vec{p},\vec{q}\right)\exp\left\{\nu\left(k^2+p^2+q^2\right)(\tau - t)\right\}d\tau$$

9. Quasi-Normal Approximation (QN)

Assume: zero forth-moment cumulants

$$<\hat{u}_n(\vec{p}')\hat{u}_m(\vec{q}')\hat{u}_j(\vec{p})\hat{u}_\ell(\vec{q})> - \left\{\begin{array}{l} <\hat{u}_n(\vec{p}')\hat{u}_m(\vec{q}')><\hat{u}_j(\vec{p})\hat{u}_\ell(\vec{q})> \\ + <\hat{u}_n(\vec{p}')\hat{u}_j(\vec{p})><\hat{u}_m(\vec{q}')\hat{u}_\ell(\vec{q})> \\ + <\hat{u}_n(\vec{p}')\hat{u}_\ell(\vec{q})><\hat{u}_m(\vec{q}')\hat{u}_j(\vec{p})> \end{array}\right\} = 0$$

Suppose isotropic and zero helicity $\equiv \frac{1}{2}<\vec{u}\cdot\vec{\omega}>=0$

$$\left(\frac{\partial}{\partial t}+2\nu k^2\right)E(k,t) = \int_0^t d\tau \iint_{\vec{p}+\vec{q}-\vec{k}=0} S(k,p,q,\tau)e^{-\nu\left(k^2+p^2+q^2\right)(t-\tau)}dpdq$$

$$S(k,p,q,\tau) = \frac{k^3}{pq}a(k,p,q)E(p,\tau)E(q,\tau) - \frac{kp^2}{pq}b(k,p,q)E(k,\tau)E(q,\tau)$$

$$a(k,p,q) \equiv \frac{1}{4k^2}P_{ijl}(\vec{k})P_{imn}(\vec{k})P_{nj}(\vec{p})P_{ml}(\vec{q})$$

$$b(k,p,q) \equiv \frac{1}{2k^2}P_{njl}(\vec{k})P_{jnm}(\vec{p})P_{ml}(\vec{q})$$

$$P_{ijl}(\vec{k}) \equiv k_l\Delta_{ij}(\vec{k}) + k_j\Delta_{il}(\vec{k})$$

9. Quasi-Normal Approximation (QN)

Orszag (1970, JFM41, 363; 1977, Fluid Dynamics, 237):

QN model leads to negative $E(k,t)$ in the energy-containing range.

The role of the forth order cumulants is to provide a damping action leading to a saturation of the third-order moment. Therefore, it cannot be ignored as did in the QN model.

Before correction:

$$\text{(9.3)} \quad \left(\frac{\partial}{\partial t}+\nu\left(k^2+p^2+q^2\right)\right)<\hat{u}_i(\vec{k})\hat{u}_j(\vec{p})\hat{u}_\ell(\vec{q})>= R_{ij\ell}(\vec{k},\vec{p},\vec{q})$$

After correction:

$$\text{(9.3)} \quad \left(\frac{\partial}{\partial t}+\nu\left(k^2+p^2+q^2\right)+\mu_{kpq}\right)<\hat{u}_i(\vec{k})\hat{u}_j(\vec{p})\hat{u}_\ell(\vec{q})>= R_{ij\ell}(\vec{k},\vec{p},\vec{q})$$

9. Eddy-Damped Quasi-Normal Approximation (EDQN)

$$\text{(9.3)} \quad \left(\frac{\partial}{\partial t}+\nu\left(k^2+p^2+q^2\right)+\mu_{kpq}\right)<\hat{u}_i(\vec{k})\hat{u}_j(\vec{p})\hat{u}_\ell(\vec{q})>= R_{ij\ell}(\vec{k},\vec{p},\vec{q})$$

μ_{kpq} : a characteristic eddy-damping rate of the third-order moments associated with the triad *(k,p,q)*

By symmetry, usually it is assumed that

$$\mu_{kpq} = \mu_k + \mu_p + \mu_q$$

Localness: $\mu_k \sim \left(k^3E(k)\right)^{1/2} \quad \left[s^{-1}\right]$

Large-eddy effect: $\mu_k \sim \left(\int_0^k p^2E(p)dp\right)^{1/2} \quad \left[s^{-1}\right]$

~ average deformation rate of eddies of size k^{-1} by larger eddies

9. Eddy-Damped Quasi-Normal Markovian Approximation (EDQNM)

Assume that the characteristic time $\left\{\nu\left(k^2+p^2+q^2\right)+\mu_{kpq}(t)\right\}^{-1}$ is much smaller than the characteristic evolution time of $R_{ij\ell}(\vec{k},\vec{p},\vec{q},t)$

(~ eddy turnover time)

$$\text{(9.3)} \quad \left(\frac{\partial}{\partial t}+\nu\left(k^2+p^2+q^2\right)+\mu_{kpq}\right)<\hat{u}_i(\vec{k})\hat{u}_j(\vec{p})\hat{u}_\ell(\vec{q})>= R_{ij\ell}(\vec{k},\vec{p},\vec{q})$$

$$<\hat{u}_i(\vec{k})\hat{u}_j(\vec{p})\hat{u}_\ell(\vec{q})>= \int_0^t R_{ij\ell}(\vec{k},\vec{p},\vec{q},\tau)e^{-\left(\nu\left(k^2+p^2+q^2\right)+\mu_{kpq}\right)(t-\tau)}d\tau$$

$$<\hat{u}_i(\vec{k})\hat{u}_j(\vec{p})\hat{u}_\ell(\vec{q})>\approx R_{ij\ell}(\vec{k},\vec{p},\vec{q},t)\int_0^t e^{-\left(\nu\left(k^2+p^2+q^2\right)+\mu_{kpq}\right)(t-\tau)}d\tau$$

$$\equiv R_{ij\ell}(\vec{k},\vec{p},\vec{q},t)\cdot\theta_{kpq}(t)$$

9. Eddy-Damped Quasi-Normal Markovian Approximation (EDQNM)

$$\theta_{kpq} \equiv \int_0^t e^{-\left(\nu\left(k^2+p^2+q^2\right)+\mu_{kpq}\right)(t-\tau)} d\tau$$

~ a characteristic time of relaxation toward a quasi-equilibrium by non-linear transfers and molecular viscosity of the third-order moments

~ If assume $\mu_{kpq} \approx \text{constant} \neq \mu_{kpq}(t)$

$$\theta_{kpq} \approx \frac{1-e^{-\left(\nu\left(k^2+p^2+q^2\right)+\mu_{kpq}\right)t}}{\nu\left(k^2+p^2+q^2\right)+\mu_{kpq}}$$

$$\rightarrow \left(\nu\left(k^2+p^2+q^2\right)+\mu_{kpq}\right)^{-1} \quad \text{as} \quad t \rightarrow \infty$$

$$<\hat{u}_i(\vec{k})\hat{u}_j(\vec{p})\hat{u}_\ell(\vec{q})> = R_{ij\ell}(\vec{k},\vec{p},\vec{q},t)\cdot\theta_{kpq}(t) = \frac{R_{ij\ell}(\vec{k},\vec{p},\vec{q},t)}{\nu\left(k^2+p^2+q^2\right)+\mu_{kpq}}$$

9. Eddy-Damped Quasi-Normal Markovian Approximation (EDQNM)

$$<\hat{u}_i(\vec{k})\hat{u}_j(\vec{p})\hat{u}_\ell(\vec{q})> = R_{ij\ell}(\vec{k},\vec{p},\vec{q},t)\cdot\theta_{kpq}(t) = \frac{R_{ij\ell}(\vec{k},\vec{p},\vec{q},t)}{\nu\left(k^2+p^2+q^2\right)+\mu_{kpq}}$$

$$\begin{aligned} R_{ij\ell}(\vec{k}) \equiv & -ik_m\Delta_{in}(\vec{k}) \iint_{\vec{p}'+\vec{q}'-\vec{k}=0} \overline{\hat{u}_n(\vec{p}')\hat{u}_m(\vec{q}')\hat{u}_j(\vec{p})\hat{u}_\ell(\vec{q})}\, d\vec{p}'d\vec{q}' \\ & -ip_m\Delta_{jn}(\vec{p}) \iint_{\vec{p}'+\vec{q}'-\vec{p}=0} \overline{\hat{u}_n(\vec{p}')\hat{u}_m(\vec{q}')\hat{u}_i(\vec{k})\hat{u}_\ell(\vec{q})}\, d\vec{p}'d\vec{q}' \\ & -iq_m\Delta_{\ell n}(\vec{q}) \iint_{\vec{p}'+\vec{q}'-\vec{q}=0} \overline{\hat{u}_n(\vec{p}')\hat{u}_m(\vec{q}')\hat{u}_i(\vec{k})\hat{u}_j(\vec{p})}\, d\vec{p}'d\vec{q}' \end{aligned}$$

$$<\hat{u}_n(\vec{p}')\hat{u}_m(\vec{q}')\hat{u}_j(\vec{p})\hat{u}_\ell(\vec{q})> = \left\{\begin{array}{l} <\hat{u}_n(\vec{p}')\hat{u}_m(\vec{q}')><\hat{u}_j(\vec{p})\hat{u}_\ell(\vec{q})> \\ +<\hat{u}_n(\vec{p}')\hat{u}_j(\vec{p})><\hat{u}_m(\vec{q}')\hat{u}_\ell(\vec{q})> \\ +<\hat{u}_n(\vec{p}')\hat{u}_\ell(\vec{q})><\hat{u}_m(\vec{q}')\hat{u}_j(\vec{p})> \end{array}\right\}$$

9. Eddy-Damped Quasi-Normal Markovian Approximation (EDQNM)

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right)\Phi_{ij} = T_{ij}$$

$$\overline{\hat{u}_i(\vec{k},t)\hat{u}_j(\vec{k}',t)} = \Phi_{ij}(\vec{k},t)\delta\left(\vec{k}+\vec{k}'\right)$$

$$T_{ij} = ik_m\left(\Delta_{jn}\Theta_{inm}(\vec{k}) - \Delta_{in}\Theta_{jnm}(-\vec{k})\right)$$

$$\Theta_{inm}(\vec{k}) \equiv \iint_{\vec{p}+\vec{q}+\vec{k}=0} \overline{\hat{u}_i(\vec{k})\hat{u}_n(\vec{p})\hat{u}_m(\vec{q})}\, d\vec{p}d\vec{q}$$

9. Other Spectral Theories

$$\frac{\partial E(k)}{\partial t} = T(k) - 2\nu k^2 E(k)$$

- Obukhoff (1941, Compt. Rend. Acad. Sci. U.R.S.S. 32, 19)

$$\int_0^k T(k')dk' = -\alpha\left(2\int_0^k k'^2 E(k')dk'\right)^{1/2}\cdot\int_k^\infty E(k')dk'$$

root-mean-square rate of strain

- Kovasznay (1948, J. Aeronaut. Sci. 15, 745)

$$\int_0^k T(k')dk' = -\alpha k^{5/2}\cdot E(k)^{3/2}$$

completely local

9. Other Spectral Theories

$$\frac{\partial E(k)}{\partial t} = T(k) - 2\nu k^2 E(k)$$

- Heisenberg (1948, Z. Physik 124, 628)

Assume: eddy (turbulent) viscosity, ν_t

$$\int_0^k T(k')dk' = -2\nu_t(k) \cdot \int_0^k k'^2 E(k')dk'$$

$$\nu_t(k) = \int_k^\infty f(k')dk'$$

$$\frac{d\nu_t}{dk} = f(k) = \alpha\sqrt{\frac{E(k)}{k^3}} \qquad \text{(localness)}$$

$$\nu_t(k) = \alpha\int_k^\infty \sqrt{\frac{E(k')}{k'^3}}dk'$$

9. Other Spectral Theories

$$\frac{\partial E(k)}{\partial t} = T(k) - 2\nu k^2 E(k)$$

- Pao(1965, Phys. Fluids 8, 1063)

$$\int_0^k T(k')dk' = -\sigma(k) \cdot E(k)$$

$\sigma(k) =$ rate that an energy spectral element is transferred across k

$$\sigma(k) = f(\bar{\varepsilon}, k) = \alpha^{-1}\bar{\varepsilon}^{1/3}k^{5/3}$$

$$\text{steady:} \quad 0 = T(k) - 2\nu k^2 E(k)$$

$$-\frac{d}{dk}\left(\alpha^{-1}\bar{\varepsilon}^{1/3}k^{5/3}E(k)\right) = 2\nu k^2 E(k)$$

$$E(k) = Ak^{-5/3}\exp\left(-\tfrac{3}{2}\alpha\nu\bar{\varepsilon}^{-1/3}k^{4/3}\right)$$

9. Other Spectral Theories

$$E(k) = Ak^{-5/3}\exp\left(-\tfrac{3}{2}\alpha\nu\bar{\varepsilon}^{-1/3}k^{4/3}\right)$$

$$\bar{\varepsilon} = 2\nu\int_0^\infty k^2 E(k)dk = \frac{A}{\alpha}\bar{\varepsilon}^{1/3} \Rightarrow A = \alpha\bar{\varepsilon}^{2/3}$$

$$E(k) = \alpha\bar{\varepsilon}^{2/3}k^{-5/3}\exp\left(-\tfrac{3}{2}\alpha\nu\bar{\varepsilon}^{-1/3}k^{4/3}\right)$$

~ Pao's spectrum

~ Obviously, α is the Kolmogorov's constant.

~ fit experimental data very well in the inertial and dissipation ranges

9. Other Spectral Theories